DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTER OF SCIENCES- MATHEMATICS

SEMESTER -IV

ALGEBRAIC TOPOLOGY

DEMATH4ELEC6

BLOCK-1

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FOREWORD

The Self Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

ALGEBRAIC TOPOLOGY

BLOCK-1

BLOCK-2

BLOCK-1 ALGEBRAIC TOPOLOGY

Introduction to block

Algebraic Topology is an important branch of topology having several connections with many areas of modern mathematics. Its growth and influence, particularly since the early forties of the twentieth century, has been remarkably high.

It is best suited for those who have already had an introductory course in topology as well as in algebra. Experience suggests that a comprehensive coverage of the topology of simplicial complexes, simplicial homology of polyhedra, fundamental groups, covering spaces and some of their classical applications like invariance of dimension of Euclidean spaces, Brouwer's Fixed Point Theorem, etc. are the essential minimum which must find a place in a beginning course on algebraic topology. Having learnt these basic concepts and their powerful techniques, one can then go on in any direction of the subject at an advanced level depending on one's interest and requirement. We introduce important examples of topological spaces in unit-1 and study the fundamental groups and its properties in Unit 2 and 3. Starting with the concept of pointed spaces we show that the fundamental groups are topological invariants of path- connected spaces. After computing the fundamental group of the circle, we show how it can be used to compute fundamental groups of other spaces by geometric methods. In chapter 4 & 5 we explain about CW- complexes and CW- homotopy

UNIT-1 IMPORTANT EXAMPLES OF TOPOLOGICAL SPACES

STRUCTURE

1.0 OBJECTIVE

In this unit we will learn about Euclidian space, spheres, Real projective spaces, Complex projective spaces, Grassmannain manifolds, Constructions, Product, Cylinder, spaces of maps, loop spaces, path spaces and Pointed spaces.

1.1 INTRODUCTION

A topological space is a set endowed with a structure, called a topology, which allows defining continuous deformation of subspaces, and, more generally, all kinds of continuity. Euclidean spaces, and, more generally, metric spaces are examples of a topological space, as any distance or metric defines a topology. The deformations that are considered in topology are homeomorphisms and homotopies. A property that is invariant under such deformations is a topological property. Basic examples of topological properties are: the dimension, which allows distinguishing between a line and a surface; compactness, which allows distinguishing between a line and a circle; connectedness, which allows distinguishing a circle from two non-intersecting circles.

1.2 EUCLIDIAN SPACE, SPHERES, DISKS

The notations R^n, C^n have usual meaning throughout the course. The space C^n is identified with R^{2n} by the correspondence

$$(x_1 + iy_1, ..., y_n + ix_n) \leftrightarrow (x_1, ..., x_n, y_n).$$

The unit sphere in R^{n+1} centered in the origin is denoted by S^n, the unit disk in R^n by D^n, and the unit cube in R^n by I^n. Thus S^{n-1} is the boundary of the disk D^n. Just in case we give these spaces in coordinates:

$$S^{n-1} = \left\{(x_1, ..., x_n) \in R^n \middle| x_1^2 + ... + x_n^2 = 1\right\},$$

(1)

$$D^n = \left\{(x_1, ..., x_n) \in R^n \middle| x_1^2 + ... + x_n^2 \le 1\right\},$$

$$I^n = \left\{(x_1, ..., x_n) \in R^n \middle| 0 \le x_j \le 1, j = 1, ..., n\right\}.$$

The symbol $x \in R^\infty$ is a sequence of points $x = (x_1, ..., x_n, ...), where\, x_n \in R\, and\, x_j = 0$ for j greater then some k.

Topology on R^∞ is determined as follows. A set $F \cap R^\infty$ is closed in R^n .In a similar way we define the spaces C^∞ and S^∞.

Exercise 1.1. Let $x^{(1)} = (a_1, 0, ..., 0, ...), ..., x^{(n)} = (0, 0, ..., a_n, ...), ...$ be a sequence of elements in R^∞. Prove that the sequence $\left\{x^{(n)}\right\}$ converges in R^∞ if and only if the sequence of numbers $\left\{a_n\right\}$ is finite.

Notes

Probably you already know the another version of infinite –dimensional real space, namely the Hilbert space ℓ_2 (which is the set of sequences $\{x_n\}$ so that the series $\sum_n x_n$ converges). The space ℓ_2 is a metric space, where the distance $p(\{x_n\},\{y_n\})$ is defined as

$$p(\{x_n\},\{y_n\}) = \sqrt{\sum_n (y_n - x_n)^2}.$$

Clearly there is a natural map $R^\infty \to \ell_2$. Remark. The optional exercises are labelled by Exercise 1.2. Is the above map $R^\infty \to \ell_2$. homeomorphism or not ? Consider the unit cube I^∞ in the spaces $R^\infty, \ell_2, i.e. I^\infty = \{\{x_n\} | 0 \le x_n \le 1\}$. Exercise 1.3. Prove or disprove that the cube I^∞ is compact space $(in\, R^\infty or\, \ell_2)$.We are going to play a little bit with the sphere S^n. Claim 1.1. A punctured sphere $S^n \setminus \{x_0\}$ is homeomorphic to R^n. Proof. We construct a map $f : S^n \setminus \{x_0\} \to R^n$ which is known as stereographic projection. Let S^n be given as above (1).Let the point x_0 be the North Pole, so it has the coordinates $(0,...,0,1) \in R^{n+1}$. Consider a point $x = (x_1,...,x_{n+1}) \in S^n, x \ne x_0$, and the line going through the points $x\, and\, x_0$. A directional vector of this line may be given as $\vec{\upsilon} = (-x_1,...,-x_n, 1 - x_n + 1)$, so any point o f this line could be written as

$$(0,...,0,1) + t(-x_1 1 - x_{n+1}) = (-tr_1,...,-tx_n, 1 + t(1 - x_{n+1})).$$

The intersection point of this line and $R^n = \{(x_1,...,x_n,0)\} \subset R^{n+1}$ is determined by vanishing the last coordinate. Clearly the last coordinate vanishes if $t = -\frac{1}{1 - x_{n+1}.}$ The map $f : S^n \setminus \{pt\} \to R^n$ is given by

$$f : (x_1,...,x_{n+1}) \leftrightarrow \left(\frac{x_1}{1 - x_{n+1}},...,\frac{x_n}{1 - x_{n+1}}, 0 \right).$$

The rest of the proof is left to you.

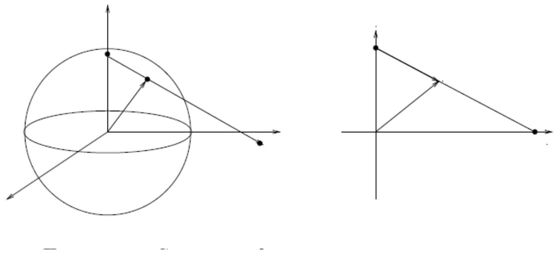

Figure 1. Stereographic projection

We define a hemisphere $S_+^n = \{x_1^2 + ... + x_{n+1}^2 - 1 \& x_{n+1} \geq 0\}$.

Exercise 1.4.Prove the that $S_+^n and\ D^n$ are homoeomorphic.

1.3 REAL PROJECTIVE SPACES

A real projective space RP^n is a set of all lines in R^{n+1} going through $0 \in R^{n+1}. Let\ \ell \in RP^n$ be a line, then we define a basis for topology on RP^n as follows:

$$U_\in(\ell) = \{\ell^1 | the\ angle\ between\ \ell\ and\ \ell^1\ less\ than \in\}$$

Exercise 1.5. A projective space RP^1 is homeomorphic to the circle S^1. Let $(x_1, ..., x_{n+1})$ be coordinates of a vector parallel to ℓ, then the vector $(\lambda x_1, ..., \lambda x_{n+1})$ defines the same line ℓ $(for\ \lambda \neq 0)$. We identify all these coordinates, the equivalence class is called homogeneous coordinates $(x_1 : ... : x_{n+1})$ .Note that there is at least one x_i which is not zero. Let

$$U_j = \{\ell = (x_1 : ... : x_{n+1}) | x_2 \neq 0\} \subset RP^n$$

Then we define the map $f_j^R : U_j \to R^n$ by the formula

$$(x_1 : ... : x_{n+1}) \to \left(\frac{x_1}{x_j}, \frac{x_2}{x_j}, ..., \frac{x_{j-1}}{x_j}, 1, \frac{x_{j-1}}{x_j}, ..., \frac{x_{n+1}}{x_i} \right).$$

Remark. The map f_j^R is a homeomorphism; it determines a local coordinate system in RP^n giving this space a structure of smooth manifold of dimension n.

There is natural map $c : S^n \to RP^n$ which sends each point $s = (s_1, ..., s_{n+1}) \in S^n$ to the line going through zero and s Note that there are exactly two points $s\ and\ -s$ which map to the same line $\ell \in RP^n$.

Notes

We have a chain of embeddings.

$$RP^1 \subset RP^2 \subset ... \subset RP^n \subset RP^{n+1} \subset ...,$$

We define $RP^\infty = U_{n\geq 1} RP^n$ with the limit topology (similarly to the above case of R^∞).

1.4 COMPLEX PROJECTIVE SPACES

Let CP^n be the space of all complex lines in the complex space C^{n+1}. In the same way as above we define homogeneous coordinates $(z_1 : ... : z_{n+1})$ for each complex line $\ell \in CP^n$, and the Local coordinate system".

$$U_i = \left\{ \ell = (z_1 : ... : z_n) \middle| z_i \neq 0 \right\} \subset CP^n.$$

Clearly there is a homomorphism $f_i^C : U_i \to C^{n+1}$.

Exercise 1.6. Prove that the projective space CP^1 is homeomorphic to the sphere S^2. Consider the sphere $S^{2+1} \subset C^{n+1}$. Each point

$$z = (z_1, ..., z_{n+1}) \in S^{2n-1},\ |z_1|^2 + ... + |z_{n+1}|^2 = 1$$

Of the sphere S^{2+1} determines a line $\ell = (z_1 : ... : z_{n+1}) \in CP^n$. Observe that the point $e^{i\varphi} z = (e^{i\varphi} z_1, ..., e^{i\varphi} z_{n+1}) \in S^{2n+1}$ determines the same complex line $\ell \in CP^n$. We have defined the map $g(n) : S^{2n+1} \to CP^n$.

Exercise 1.7. Prove that the map $g(n) : S^{2n+1} \to CP^n$ has a property that $g(n)^{-1}(\ell) = S^1 \text{ for any } \ell \in CP^n$.

The case n=1 is very interesting since $CP^1 = S^2$, here we have the map $g(1) : S^3 \to S^2$ where $g(1)^{-1}(x) = S^1$ for any $x \in S^2$. This map is the Hopf map, it gives very important example of nontrivial map $S^3 \to S^2$.

Exercises 1.8. Prove that RP^n, CP^n are compact and connected spaces.

Besides the reals R and complex numbers C there are quaternion numbers H. Recall that $q \in \mathbf{H}$ may be thought as a sum $q = a + ib + jc + kd, \textit{where}\ a, b, c, d \in R,$ and the symbols i, j, k satisfy the identifies:

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

Then two quaternions $q_1 = a_1 + ib_1 + jc_1 + kd_1 \; and \; q_2 = a_2 + ib_2 + jc_2 + kd_2$ may be multiplied using these identities. The product here is not commutative, however one choose left or right multiplication to define a line in H^{n+1}. A set of all quaternionic lines in H^{n+1} is the quaternion projective space $\mathbf{HP}^n$.

Exercise 1.9. Give details of the above definition. In particular, check that the space $\mathbf{HP}^n$ is well-defined. Identify the quaternionic line $\mathbf{HP}^1$ with some well-known topological space.

1.5 GRASSMANNIAN MANIFOLDS

These spaces generalize the projective spaces. Indeed, the space $G(n,k)$ is a space of all k-dimensional vector subpaces of R^n. with natural topology. Clearly $G(n,1) = RP^1$. It is not difficult to introduce local coordinates in $G(n,k)$. Let $n \in G(n,k)$ be a k-plane. Choose k linearly independent vectors $v_1,...,e_n of \; R^n$:

$$A = \begin{pmatrix} a_{11} & ... & a_{1n} \\ . & . & . \\ . & . & . \\ . & . & . \\ a_{k1} & ... & a_{kn} \end{pmatrix}$$

Since the vectors $\upsilon_1,...,\upsilon_k$ are linearly independent there exist k columns of the matrix A which are linearly independent as well. In other words, there are indices $\upsilon_1,...,\upsilon_k$ so that a projection of the plane π on the $k - plane \langle e_{i1},...,e_{ik} \rangle$ generated by the coordinate vectors $e_{i1},...,e_{ik}$ is a linear isomorphism. Now it is easy to introduce local coordinates on the Grassmanian manifold indeed, choose the indices $i_1,...,i_k, 1 \le i_1 < ... < i_k \le n,$ and consider all $k - planes \, \pi \in G(n,k)$ so that the projection of π on the plane $(e_{i1},...,e_{ik})$ is a linear isomorphism. We denote this set of $k - planes \, by \, U_{i1},...,i_k$.

Exercise 1.10. Construct a homeomorphism $f_{i1,...,ik} : U_{i1,...,ik} \rightarrow R^{k(n-k)}$.

The result of this exercise shows that the Grassmannian manifold $G(n,k)$ is a smooth manifold of dimension $k(n-k)$. The projective spaces and grassmannian manifolds are very important examples of spaces which we will see many times in our course.

Exercise 1.11. Define a complex Grassmannian manifold $CG(n,k)$ and construct a local coordinate system for $CG(n,k)$. In Particular, find its dimension.

We have a chain of spaces

$G(k,k) \subset (k+1,k) \subset ... \subset G(n,k) \subset G(n\_1,k) \subset ....$

Let $G(\infty,k)$ be the union (inductive limit) of these spaces. The topology of $G(\infty,k)$ is given in the same way as to $R^{\infty}: a\,set\,F \subset G(\infty,k)$ is closed if and only if the intersection $F \bigcap G(n,k)$ is closed for each n. This topology is known as a topology of an inductive limit.

Exercise 1.12. Prove that the Grassmannian manifolds $G(n,k)$ and $CG(n,k)$ are compact and connected.

1.5 Flag manifolds. Here we just mention these examples without further considerations (we are not ready for this yet). Let $1 \le k_1 < ... < k_s \le n-1$. A flag of the type $(k_1,...,k_s)$ is a chain of vector subspaces $V_1 \subset ... \subset V_s\ of\ R^n$ such that dim $V_1 = k_i$. A set of flags of the given type is the flag manifold $F(n;k_1,....k_s)$. Hopefully we shall return to these spaces they are very interesting and popular creatures.

Classic Lie groups. The first example here is the group $GL(R^n)$ of non generated linear transformations of R^n. Once we choose a basis $e_1,...,e_n\ of\ R^n$, each element $A \in GL(R^n)$ may be identified with an $n \times n$ matrix. A with det $A \ne 0$. Clearly we may identify the space of all $n \times n$ matrices with the space R^{n2}. The determinant gives a continuous function $\det : R^{n2} \to R$, and the space $GL(R^n)$ is an open subset of R^{n2}.

$$GL(R^n) = R^{n^2} \setminus \det^{-1}(0).$$

We note that the group $O(n)$ acts on the spaces $V(n,k)\ and\ G(n,k)$:indeed, if

$\alpha \in O(n)$ and $\upsilon_1,...,\upsilon_k$ is an orthonormal $k-\ frame,$ then $\alpha(\upsilon_1),...\alpha(\upsilon_k)$ is also an orthonormal $k-\ frame.$ As for the Grassmannian manifold, one can easily see that $\alpha(\prod)$ is a K-dimensional subspace in $O(k) \times O(n-k)$

The group $O(n)$ contains a subgroup $O(j)$ which acts on $R^j \subset R^n$, where $R^j, \langle e_1,...,e_j \rangle$ is generated by the first j vectors $e_1,...,e_j$ of the standard basic $e_1,...,e_n$ of R^n.Similarly $U(n)$ act on the spaces $CG(n,k)$ and $GV(n,k), and\, U(j)$ is a subgroup of $U(n)$.

Exercise 1.20. Prove the following homeomorphisms:

(a) $S^{n-1} \cong O(n)/O(n-1) \cong SO(n)/SO(n-1),$

(b) $S^{n-1} \cong U(n)/U(n-1) \cong SU(n)/SU(n-1),$

(c) $G(n,k) \cong O(n)/O(k) \times O(n-k),$

(d) $CG(n,k) \cong U(n)/U(k) \times U(n-k).$

We note here that $O(k) \times O(n-k)$ is a subgroup of $O(n)$ of orthogonal matrices with two diagonal blocks of the sizes $k \times k\; and\; (n-k) \times (n-k)$ and zeros otherwise.

There is also the following natural action of the orthogonal group $O(k)$ on the Stieffel manifold $V(n,k)$,Let $(\upsilon_1,...,\upsilon_k)$ be on orthogonal $k-frame.$ then $O(k)$ acts on the space $V = \langle \upsilon_1,...,\upsilon_k \rangle$,in particular if $\alpha(\upsilon_1),...,\alpha(\upsilon_k)$ is also an orthogonal $k = frame$ similarly there is natural action of $U(k)$ on $CV(n,k)$.

Exercise 1.21 Prove that the above actions of $O(k)$ on $V(n,k)$ and of $U(k)$ on

$CV(n,k)$ are free.

Exercise 1.22 Prove the following homeomorphisms;

(a) $V(n,k)/O(k) \cong G(n,k),$

(b) $CV(n,k)/U(k) \cong CG(n,k).$

There are obvious maps $V(n,k) \xrightarrow{p} G(n,k), CV(n,k) \xrightarrow{p} CG(n,k)$ (where each orthonormal $k = frame\; \upsilon_1,...,\upsilon_k$ maps to the $k = plane\; \pi = \langle \upsilon_1,...,\upsilon_k \rangle$ generated by this frame).It is easy to see that the inverse image $p^{-1}(\pi)$ may be identified with $O(k)$ (in the real case) and $U(k)$ (in the complex case). We shall return to these space later on. In particular, we shall describe a cell-structure of these spaces and compute their homology and cohomology groups.

Notes

Surfaces. Here I refer to Chapter 1 of Massey, Algebraic topology, for details. I would like for you to read this Chapter carefully even though most of you have seen this material before. Here I briefly remind some constructions and give exercises. The section 4 of the reffered Massery book gives the examples of surfaces. In particular, the torus T^2 is described in three different ways.

(a) *A product* $S^1 \times S^1$.

(b) A subspace of R^3 given by :

$\left\{(x, y, z) \in R^3 \middle| (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\right\}.$

(c) A unit square $I^2 = \left\{(x, y) \in R^2 \middle| 0 \le x \le 1, 0 \le y \le 1\right\}$ with the identification:

$(x, 0) \equiv (x, 1)$ $(0, y) \equiv (1, y)$ *for all* $0 \le x \le 1, \quad 0 \le y \le 1.$

Exercise 1.23. Prove that the space described in $(a), (b), (c)$ are indeed homeomorphic.

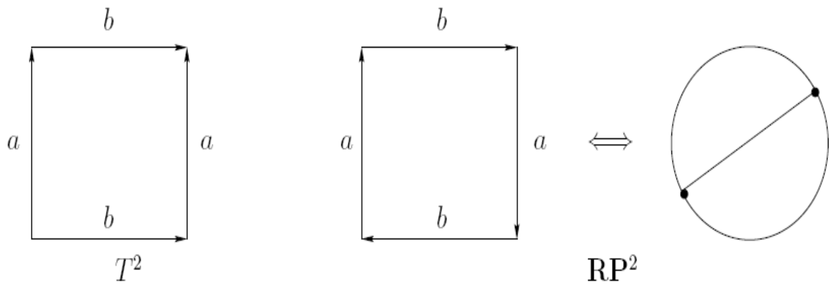

FIGURE 2. Torus and projective plane

The next surface we want to become our best friend is the projective space RP^2. Earlier we defined $D^2 = \left\{(x, y) \in R^2 \middle| x^2 + y^2 \le 1\right\}$ as a space of lines in R^3 going through the origin.

Exercise 1.24. Prove that the projective plane RP^2 is homeomorphic to the following spaces:

(a) The unit disk $D^2 = \{(x,y) \in R^2 \,|\, x^2 + y^2 \le 1\}$ with the opposite points $(x,y) \equiv (-x,-y)$ of the circle $S^1 = \{(x,y) \in R^2 \,|\, x^2 + y^2 \le 1\} \subset D^2$ have been identified

(b) The unit square, see fig.3, with the arrows a and b identified as it is shown.

(c) The Mebius band which boundary (the circle) is identified with the boundary of disk

$D^2, see\, Fig.3.$

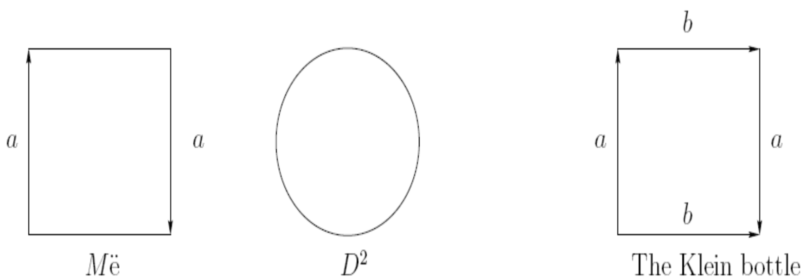

FIGURE 3

Here the Mebius band is constructed from a square by identifying the arrows a. The Klein bottle Kl^2 may be described a s a square with arrows identified as it is shown in Fig.3. Exercise 1.25. Prove that the Klein bottle Kl^2 is homeomorphic to the union of two Mebius bands along the circle. Massey carefully defines connected sum $S_1 \# S_2$ of two surfaces $S_1 and\, S_2$.

Exercise 1.26. Prove that $Kl^2 \# RP^2$ is homeomarphic to $RP^2 \# T^2$

Exercise 1.27. Prove that $K1^2 \# Rl^2$ is homeomarphic to Kl^2.

Exercise 1.28. Prove that $RP^2 \# RP^2$ is homeomarphic to Kl^2.

1.6 CONSTRUCTIONS

1.6.1 Product

Recall that a product $X \times Y\, of\, X, Y$ is a set of pairs $(x,y), x \in X, y \in Y. If\, X, Y$ are topological spaces then a basis for product topology on $X \times Y$ is given by the products $U \times V, where U \subset X, V \subset Y$ are open. Here are the first examples.

Notes

Example. The torus $T^n = S^1 \times ... \times S^1$. Note that the torus T^n may be identified with $U(1) \times ... \times U(1) \subset U(n)$ (diagonal orthogonal complex matrices).

Exercise 1.7 Consider the surface $X \, in \, S^5$, given by the equation

$$x_1 x_0 - x_2 x_5 + x_3 x_4 = 0$$

(where $S^5 \subset R^5$ is given by $x_1^2 + ... + x_6^2 = 1$). Prove that $X \cong S^2 \times S^2$.

Exercise Prove that the space $SO(4)$ is homeomorphic to $S^3 \times RP^3$.

Hint: Consier carefully the map $SO(4) \to S^3 = SO(4)/SO(3)$ and use the fact that S^3 has natural group structure: it is a group of unit quaternions .It should be emphasized that it is not true that $SO(n) \cong S^{n-1} \times SO(n-1) \, if \; n > 4.$

We note also that there are standard projections $X \times Y \xrightarrow{y \in X} X \; and \; X \times Y \xrightarrow{p \in Y} Y$, and to give a map $f : Z \to X \times Y$ $f : Z \to X \times Y$ is the same as to give two maps $f_x : Z \to X \; and \; f_y : Z \to Y.$

1.6.2 Cylinder, suspension

Let $I = [0,1] \subset R$, The space $X \times I$ is called a cylinder over X, and the subspaces $X \times \{0\}, X \times \{1\}$ are the bottom and top "bases". Now we will construct new spaces out of cylinder $X \times I$..

Remark: quotient topology. Let $"R"$ be an equivalence relation on the topological space X. We denote by X/R the set of equivalence classes. There is a natural map (not continuous so far) $p : X \to X/\cong$. We define the following topology on X/R: the set $U \subset X/R$ is open if and only if $P^{-1}(U)$ is open. This topology is called a quotient topology.

The first example: let $A \subset X$ be a closed set. Then we define the relation $"R"$ on X as follows $\left([\;] \, denote \, an \, equivalence \, class \right)$:

$$[x] = \begin{cases} \{x\} & if \; x \notin A, \\ A & if \; x \in A. \end{cases}$$

The space $X/\square$ is denoted by X/A.

The space $C(X) = X \times 1 / X \times \{1\}$ is a cone over X. A suspention $\sum S^n$ are homeomorphic to D^{n+1} and S^{n+1} respectively.

Here is a picture of these spaces.

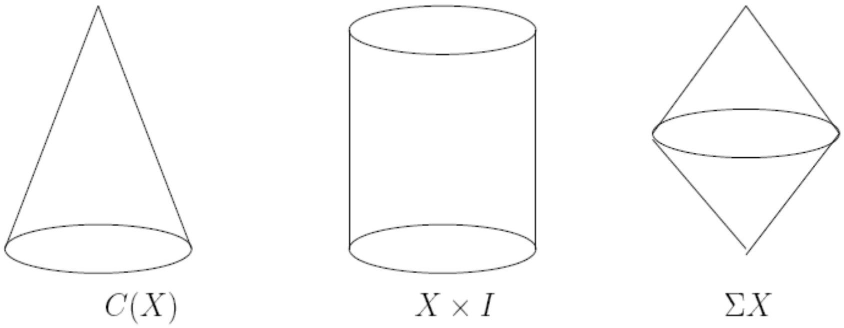

FIGURE 4

Glueing: Let X *and* Y be topological spaces, $A \subset Y$ *and* $\varphi : A \to X$ be a map. We consider a disjoint union $X \subset Y$, and then we identify a point $a \in A$ with the point $\varphi(a) \in X$. The quotient space $X \cup Y / R$ under this identification will be denoted as $X \cup_{\varphi} Y$, and this procedure will be called glueing X *and* Y by means of φ. There are two special cases of this construction.

Let $f : X \to Y$ be a map. We identify X with the bottom base $X \times \{0\}$ of the cylinder $X \times 1$. The space $X \times 1 \cup_f Y = Cyl(f)$ is called a cylinder of the map f. the space $C(X) \cup_f Y$ is called a cone of the map f. Note that the space $Cyl(f)$ contains X *and* Y as subspaces, and the space $C(f)$ contains X.

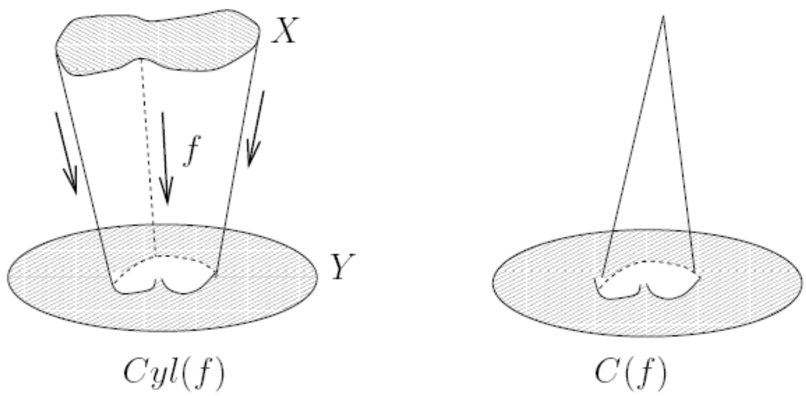

FIGURE 5

Let $f : S^n \to RP^n$ be the (we have studied before) map which takes a vector $\vec{\upsilon} \in S^n$ to the line $\ell = \langle \vec{\upsilon} \rangle$ spanned by $\vec{\upsilon}$.

Notes

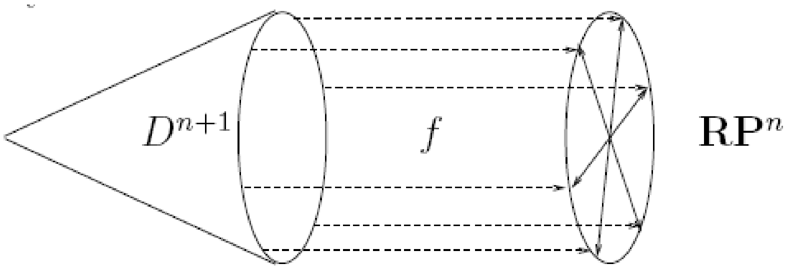

FIGURE 6

Claim: The cone $C(f)$ is homeomorphic to the projective space RP^{n+1}.

Proof (outline). Consider the cone over $S^n, clearly\, C(S^n) \cong D^{n+1}$

Now the cone $C(f)$ is a disk D^{n+1} with the opposite points of S^n identified. See Fig. 6.

In particular, a cone of the map $f: S^1 \to S^1 = RP^1$ (given by the formula $e^{i\varphi} \mapsto e^{2i\varphi}$) coincides with the projective plane $RP^{2.}$

Exercise: Prove that a cone $C(h)$ of the Hopf map $h: S^{2n+1} \to CP^n$ is homeomorphic to the projective space CP^{n+1}.

Here is the construction which should help you with previous exercise .Let us take one more look at the Hopf map $h: S^{2k+1} \to CP^k:$ we take a point $(z_1,...,z_{k+1}) \in S^{2k+1}, (where |z_1|^2 + ... + |z_{k+1}|^2 = 1)$, then h takes it to the line $(z_1 : ... : z_{k+1}) \in CP^k$. Moreover $h(z_1 ... z_{k+1}) = (z_1^{'},...,z_{k+1}^{'})$ if and only if $z_j^{'} = e^{i\varphi} z_j$. This we can identify CP^k with the following quotient space:

(2) $CP^k S^{2k+1} / R$, where $(z_1^{'},...,z_{k+1}^{'}) R (e^{i\varphi} z_{1,} ..., e^{i\varphi} z_{k+1}^{'})$.

Non consider a subset of lines in CP^k where the last homogeneous coordinate is nonzero:

$$U_{k+1} = \{(z_1 : ... : z_{k+1}) | z_{k+1} \neq 0\}.$$

We already know that U_{k+1} is homeomorphic to C^k by means of the map

$$(z_1 : ... : z_{k+1}) \mapsto \left(\frac{z_1}{z_{k+1}}, ..., \frac{z_1}{z_{k+1}} \right)$$

Now we use (2) to identify U_{k+1} with an open disk $D^{2k} \subset C^k$ as follows. Let us think about $U_{k+1} \subset S^{2k+1}/R$ as above. Let $\ell \in U_{k+1}$. Choose a point $(z_1 : ... : z_{k+1}) \in S^{2k+1}$ representing ℓ. Then we have that

$$\left|z_1\right|^2 + ... + \left|z_k\right|^2 = 1 = r^2$$

Which describes the sphere $S^{2k-1}\sqrt{1+r^2} \subset C^k$ of radius $\sqrt{1-r^2}$. The union of the spheres $S^{2k-1}\sqrt{1+r^2}$ over $0 < r \leq 1$ is nothing but an open unit disk in C^k. Then we notice that we can let z_{k+1} to be equal to zero: $z_{k+1} = 0$ corresponds to the points

$$(z_1, ..., z_k, 0) \in S^{2+1} with \left|z_k\right|^2 + ... + \left|z_k\right|^2 = 1,$$

i.e. the sphere $S^{2+1} \subset C^k$ modulo the equivalence relation $(z_1, ..., z_{k,} 0) R (e^{i\varphi} z_1, ..., e^{i\varphi} z_k, 0)$. This is nothing but eh projective space CP^{k-1}. We summarize our construction:

lemma2.1. There is a homeomorphism

$$CP^k \equiv D^{2k}/R,$$

Where $(z_1, ..., z_k) R (z_j^{'}, ..., z_j^{'})$ *if and only if*

$$\left\{ \left|z_1\right|^2 + ... + \left|z_k\right|^2 = 1, \left|z_1^{'}\right|^2 + ... + \left|z_1^{'}\right|^2 = 1, and \right.$$

$$z_j^{'} = e^{i\varphi} z_j \qquad \qquad for\ all\ j = 1, ..., k.$$

join. A join $X * Y$ of spaces $X\ Y$ is a union o fall linear paths $I_{x,y}$ starting at $x \in X$ and ending at $y \in Y$; the union is taken over all points $x \in X$ and $y \in Y$. For example, a joint of two intervals $I_1\ and\ I_2$ lying on two non-parallel and non-intersecting lines is terrahedron: A formal definition of $X * Y$ is the following. We start with the product

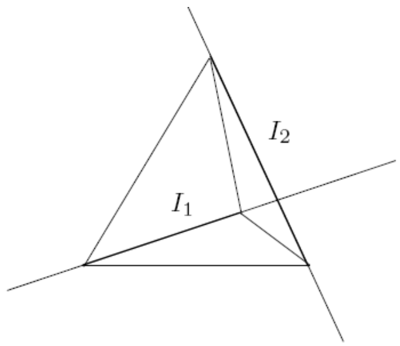

FIGURE 7

$X \times Y \times I$: here there is a linear path $(x, y, t), t \in I$ for given points $x \in X, y \in Y$. Then we identify the following points:

$$(x, y^1, 1)R(x, y^n, 1) \text{ for any } x \in Y, y^1, y^n \in Y,$$

$$(x^{'}, y, 0)R(x^{''}, y, 0) \text{ for any } x^{'}, x^{''} \in X, y \in Y.$$

Exercises 2.5 prove the homeomorphisms

$(a) X * \{one\ po\text{int}\} \cong C(X):$

$(b) X * \{two\ po\text{int}\, s\} \cong \sum(X):$

$(c) S^n * S^k \cong S^{n+k+1}. H\text{ int}: prove\ first\ that\ S^1 * S^1 \cong S^3.$

1.6.3 Spaces of maps, loop spaces, path spaces

Let X, Y are topological spaces. We consider the space $C(X,Y)$ of all continuous maps from $X\ to Y$. To define a topology of the functional space $C(X,Y)$ it is enough to describe a basis. The basis of the compact-open toplogy is given as follows. Let $K \subset X$ be a compact set, and $O \subset Y$ be an open set. We denote by $U(K,O)$ the set of all continuous maps $f : X \to Y$ such that $f:(K) \subset Y$, this is (by definition) a basis for the compact-open topology on $C(X,Y)$.

Examples. Let X be a point, Then the space $C(X,Y)$ is homeomorphic to $Y, If\ X$ be a space consisting of n points, then $C(X,Y) \cong Y \times ... \times Y\ (n\, times)$.

Let $X, Y, and\ Z$ be Hausdorff and locally $compact$[1] topological spaces. There is a natural map

$$T : C(X, C(Y, Z)) \to C(X \times Y, Z),$$

Given by formula: $\{f : X \to C(Y, Z)\} \to \{(x, y) \to (f(x))(y)\}$.

Exercise: Prove that the map $T : C(X, C(Y, Z)) \to C(X \times Y, Z)$ is a homeomorphism.

[1] A topological space X is called locally compact if for each point $x \in X$ and an open neighbourhood $U\ of\ X$ there exists an open neighbourhood $V \subset U$ such that the closure $\vec{V}\ of\ V$ is compact.

Recall we call a map $f : I \to X$ a path, and the points $f(0) = f(1) = x_1$ are the beginning and the end points of the path f. The space of all paths $C(I, X)$ contains two important subspaces:

1. $\varepsilon(X, x_0, x_1)$ is the subspace of paths $f : I \to X$ such that $f(0) = x_0 and\ f(1) = x_1$;

2. $\varepsilon(X, x_0)$ is the subspace of all paths with x_0 the beginning point.

3. $\Omega(X, x_0) = \varepsilon(X, x_{0,} x_0)$ is the loop space with the beginning point x_0.

Exercise: Prove that the spaces $\Omega(S^n, x) and\ \Omega(S^n, x^{'})$ are homeomorphic for any points $x, x^{'} \in S^n$.

Exercise: Give examples of a space X other than S^n for which $\Omega(X, x)$ and $\Omega(X, x^{'})$ are homeomorphic for any points $x, x^{'} \in X$. Why does it fail for an orbitrary space X? Give an example when this is not true.

The loop spaces $\Omega(X, x)$ are rather difficult to describe even in the case of $X = S^n$, however, the spaces $X\ and\ \Omega(X, x)$ are intimately related. To see that, consider the following map (3)

$$p : \varepsilon(X, x_0) \to X$$

Notes

Which sends a path $f: I \to X, f(0) = x_0,$ to the point $x = f(1).$ Notice that $p^{-1}(x_o) \cong \Omega(X, x_0).$ The map (3) may be considered as a map of pointed spaces (se the definitions below):

$$p: (\varepsilon(X, x_0), *) \to (X, *),$$

Where the path $*: I \to X$ sends the interval to the point $*(t) = x_0$ for all $t \in I.$ Clearly $p(*) = x_0.$

1.6.4 Pointed spaces

A pointed space (X, x_0) is a topological space X together with a base point $x_0 \in X.$ A map $f: (X, x_0) \to (Y, y_0)$ is a continuous map $f: X \to Y$ such that $f(x_0) = y_0.$ Many operations preserve base points, for example the product $X \times Y$ of pointed spaces (X, x_0), (Y, y_0) have the base point $(x_0, y_0) \in X \times Y.$ Some other operations have to be modified.

The cone $C(X, x_0) = C(X) / \{x_0\} \times I:$ here we identify with the point all interval over the base point $x_{0,}$ and the image of $\{x_0\} \times I \, in C(X, x_0)$ is the base point of this space.

The suspension:

$$\sum(X, x_0) = \sum(X) / \{x_0\} \times I = C(X) / (X \times \{0\} U x_0 \times I) = C(X, x_0) / (X \times \{0\}).$$

The space of maps $C(X, x_0, Y, y_0)$ for pointed spaces $^2(X, x_0)$ and (Y, y_0) is the space of continuous maps $f: X \to Y$ such that $f: (x_0) = y_0$ (with the same compact-open topology). The base point in the space $C(X, Y)$ is the map $c: X \to Y$ which sends all space X to the point $y_0 \in Y.$

If X is a pointed space, then $\Omega(X, x_0)$ is the space of loops beginning and ending at the bas point $x_0 \in X,$ and the space $\varepsilon(X, x_0)$ is the space of paths starting at the base point $x_{0,}$

Check Your Progress

1. Prove: The cone is homeomorphic to the projective space

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2. Explain about space, spheres and disks.

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3. Explain about Real projective spaces.

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1.7 LET US SUM UP

1. The notations R^n, C^n have usual meaning throughout the course. The space C^n is identified with R^{2n} by the correspondence

$$(x_1 + iy_1, ..., y_n + ix_n) \leftrightarrow (x_1, ..., x_n, y_n).$$

2. A real projective space RP^n is a set of all lines in R^{n+1} going through $0 \in R^{n+1}. Let\ \ell \in RP^n$ be a line, then we define a basis for topology on RP^n as follows:

$$U_{\in}(\ell) = \left\{ \ell^1 \middle| the\ angle\ between\ \ell\ and\ \ell^1\ less\ than \in \right\}$$

3. Let "R" be an equivalence relation on the topological space X. We denote by X/R the set of equivalence classes. There is a natural map (not continuous so far) $p : X \to X/\cong$. We define the following topology on X/R: the set $U \subset X/R$ is open if and only if $P^{-1}(U)$ is open. This topology is called a quotient topology.

4. A pointed space (X, x_0) is a topological space X together with a base point $x_0 \in X$. A map $f:(X, x_0) \to (Y, y_0)$ is a continuous map $f: X \to Y$ such that $f(x_0) = y_0$.

1.8 KEY WORDS

Euclidian space

Real projective spaces

Complex projective spaces

Topology

Quotient topology

1.9 QUESTIONS FOR REVIEW

1. Explain about Complex projective spaces.
2. Explain about Grassmananian manifolds.
3. Explain about Space of maps, loop spaces, path spaces.

1.10 SUGGESTIVE READINGS AND REFERENCES

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1.11 ANSWERS TO CHECK YOUR PROGRESS

1. See section 1.2
2. See section 1.2
3. See section 1.3

UNIT-2 THE FUNDAMENTAL GROUP

STRUCTURE

2.0 OBJECTIVE

In this unit we will learn and understand about homotopy, Contractible spaces and homotopy type, and important definitions and theorems and related exercise questions.

2.1 INTRODUCTION

Let X be a topological space. Often we associate with X an object that depends on X as well as on a point x of X. The point x is called a base point and the pair (X,x) is called pointed space. If (X,x) and (Y,y) are two pointed spaces, then a continuous map $f:X\to Y$ such that $f:(x)-y$ is called a map between pointed spaces. Let $f:X\to Y$ $f:(x)=y$ be a homeomorphism and x be a point of X. Then f is a homeomorphism between pointed spaces. (X,x) and $\left(Y,f(x)\right)$. The composite of two maps between pointed spaces is again a map between pointed spaces and the identity map $I_{(X,x)}:(X,x)\to(X,x)$ is always a homeomorphism of pointed spaces for each $x\in X$.

In this chapter, we show low to each pointed space (X,x), we can associate a group (X,x), called the fundamental group of the space X at x. Each map $f:(X,x)\to(Y,y)$ between pointed spaces $(X,x)\, and\, (Y,y)$ then induces a homomorphism (denoted by $f_{\#}$ also).

$f_*:\pi_1(X,x)\to\pi_1(Y,y)$

Between groups $\pi_1(X,x)\, and\, \pi_1(Y,y)$ such that the following two conditions are satisfied:

(i) If $f:(X,x)\to(Y,y) and\; g:(Y,y)\to(Z,z)$ are two maps of pointed spaces, then

$(g\circ f)_*=g_*\circ f_*:\pi_1(X,x)\to\pi_1(Z,z).$

(ii) If $I_{(X,x)}:(X,x)\to(X,x)$ is the identify map, then the induced gropu homomorphism

$I(X,x)_*:\pi_1(X,x)\to\pi_1(X,x)$

Is also the identify map.

We will also compute fundamental group of several interesting spaces and exhibit some of their uses. The two properties of the induced homomorphism stated above are known as functional properties, which at once yield the following important consequence: the fundamental group $\pi_1(X,x)$ is a topological invariant of the pointed space (X,x). The detailed meaning of this statement is given below:

Proposition 2.1.1. If (X,x) and (Y,y) are two pointed spaces which are homeomorphism,then their fundamental groups $\pi_1(X,x)\, and\, \pi_1(Y,y)$ are isomorphic.

Proof: Suppose $f:(X,x)\to(Y,y)$ is a homeomorphism of pointed spaces. Then the inverse map $f^{-1}:(Y,y)\to(X,x)$ is evidently a map of pointed spaces and has the property that

$$f^{-1}\circ f=I_{(X,x)},\ f\circ f^{-1}=I_{(Y,y).}$$

Let us consider the induced homeomorphisms

$$f_*:=\pi_1(X,x)\to\pi_1(Y,y),\ f_*^{-1}:\pi_1(Y,y)\to\pi_1(X,x)$$

In the fundamental groups. By functional property (i), we find that

$$(f^{-1}\circ f)_*=f^{-1}{}_*\circ f_*,$$

And, by (ii),we see that $I(X,x)$ is the identity map on $\pi_1(X,x).$ Thus, we conclude that $f\ and\ f^{-1}$ is the identity map on $\pi_1(X,x).$ Similarly, we can see that $f\ and\ f^{-1}$ is the identity map on Therefore, $f\ and\ f^{-1}$ are inverses of each other, i.e., $f:\pi_1(X,x)\to\pi_1(Y,y)$ is an isomorphism.

The fact that the fundamental group $\pi_1(X,x)$ of a pointed space (X,x) of a pointed space (X,x) is a topological invariant is an interesting and a very useful result. It says that if X and Y are two spaces such that for some $x_0\in X,\pi_1(X,x_0)$ is not isomorphic to any of $\pi_1(Y,y),y\in Y,$ then the spaces X and Y cannot be homoeomorphic. For, suppose X and Y are homoeomorphic and let $f:(X,x)\to(Y,y)$ be a homeomorphism. Then $f:(X,x_0)\to(Y,f(x_0)$ is a homeomorphic of pointed spaces and so, by the above proposition, the induced group homomorphic $f:\pi_1(X,x_0)\to\pi_1(Y,f(x_0))$ must be an isomorphism, i.e., $\pi_1(X,x_0)$ is isomorphic to $\pi_1(y,f(x_0)),$ a contradiction.

One of the most important problems in topology, known as the classification problem in a given class of topological spaces, is to decide whether or not two given spaces of that class are homeomorphic. To prove that two spaces X and Y are indeed homeomorphic, the problem is really to find out some specific homeomorphism from X and Y, and invariably, the only method to do this is the knowledge of point-set topology. However, to prove that X and Y are not homeomorphic, one looks for some topological invariant possessed by one space and not by the other. For example, when we have to show that R^1 is not homeomorphic to R^2, we argue as follows: if we remove one point from both, then the remaining spaces, first being disconnected and the second being.

Connected, are not homeomorphic and so R^1cannot be homeomorphic to R^2.Similarly, the circle S^1cannot be homeomorphic to the figure of eight (two circles touching at a point) because if we remove the point of contact from the figure of eight, then the remaining space is disconnected whereas if we remove any point from S^1, the remaining space remains connected. To prove that a closed interval $[0,1]$ is not homeomorphic to an open interval (0,1), we say that one is compact whereas the other is not compact and so they cannot be homeomorphic. These methods are known as the methods of pint set topology. Now, let us ask whether or not the 2-sphere $\pi_1(X,x)$ is homeomorphic to the 2-torus $T = S^1 \times S^1$ .This can also be resolved using point-set topology as follows: take a circle $C\, in T$ as shown in figure.

If we remove $C\, from T$, the remaining space is clearly connected. However, if we remove any circle from S^2, the remaining space is evidently disconnected. This says that T and S^2can not be homeomorphic. Now, let us ask whether or not the 3-sphere $S^3$is homeomorphic to the 3-torus $T = S^1 \times S^1 \times S^1$.

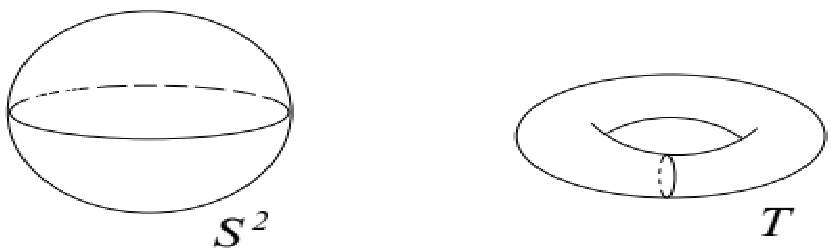

Fig. 2.1: A 2-sphere and a 2-torus

The reader is invited to discover some method of point-set topology to show that they are not homoeomorphic (they are really not homoeomorphic) and see for himself that this can be extremely difficult. In such a case, however, the methods of algebraic topology sometimes work very well. We will, later on, prove that if X is a path-connected space then the fundamental group $\pi_1(X,x)$ is independent of the base point (up to isomorphism),and we denote it simply by $\pi_1(X)$.This fact, combined with the previous proposition, will say that if $X\ and\ Y$ are two path-connected spaces such that $\pi_1(X)$ is not isomorphic to $\pi_1(Y)$ then $X\ and\ Y$ cannot be homoeomorphic. By popular belief based on experience, it is normally much easier to decide that two groups are not isomorphic than to decide that two given spaces are not homoeomorphic. Now, granting that $\pi_1(S^3)=0\ and\ \pi_1(S^1\times S^1\times S^1)\cong Z\times Z\times Z$ (we shall prove these facts later), we immediately conclude that S^3 cannot be homoeomorphic to the torus $S^1\times S^1\times S^1$ as their fundamental groups are evidently not isomorphic. This is just one of the several methods of algebraic topology in proving that two paces are not homoeomorphic. The crucial point is the result that the fundamental group $\pi_1(X)$ is a topological invariant of path connected spaces. Several objects such as homology groups. Euler characteristics, etc., are other important invariants of topological spaces. We have only indicated that fundamental group $\pi_1(X,x)$ of a pointed space (X,x) is a topological invariant. This is a good result, but by no means the best result. More general and interesting results, including the best ones about fundamental groups, will be studied only after we have defined them. The definition requires the important concept of homotopy and the generalization of results will require the concept of homotopy type in the class of topological spaces. We take up these notions in the next section.

2.2 HOMOTOPY

We now proceed to define fundamental group $\pi_1(X,x)$ of a given pointed space (X,x). It will take some time to do so, but that is true of most of the topological invariants in algebraic topology. The fundamental group is, of course, the first such invariant we are going to deal with. There are several important concepts which will be introduced on our way to the definition of $\pi_1(X,x)$.The first one is the concept of homotopy. We have

Definition 2.3.1.Let X,Y be two spaces and $f,g:X\to Y$ be two continuous maps. We say that f is homotopic to g $(and\ denote\ it\ by\ writing\ f\cong g)$ if there exists a continuous map $F:X\times I\to Y$ such that $F(x,0)-f(x)$ and $F(x,1)=g(x)\ for\ all\ x\in X$. The map F is called a homotopy from $f\ to\ g$.

We know that for each $t\in I$, the map $i_t:X\to X\times 1$ defined by $i_t:(x)=(x,t)$ is an embedding. So, $f_t=F\,o\,i_t:X\to Y$ is a family of continuous maps from $X\ to\ Y$, where t runs over the interval I. By the definition of homotopy F, f_t is the map $f\ for\ t=0\ and\ for\ t=1$ it is the map g. Thus, a homotopy F is simply a family of continuous maps from $X\ and\ Y$,where t urns over the interval I. By the definition of homotopy F is simply a family of continuous maps from $X\ and\ Y$ which starts from f ,changes continuously with respect to t and terminates into the map g. In other words, f gets continuously transformed by means of the homotopy F and finally changes or deforms itself into the map g. See Fig.

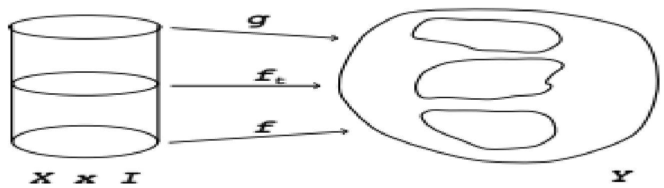

Fig. 3.2: F is a Homotopy from $f=F\circ i_\circ to\, g=F\circ i_1$

Notes

Example 2.3.2. Let $X = R^n = Y$ be the Euclidean spaces and let $f, g : X \to Y$ be defined by $f(x) = x \, and \, g(x) = 0, x \in X$. Define the map $H : R^n \times I \to R^n$ by

$$H(x,t) = (1-t)x.$$

Then, clearly, H is continuous and for all $x \in X, H(x,0) = x = (f(x), H(x,1) = g(x). Thus, H$ is a homotopy from $f \; to \; g \; and \; so \; f$ is homotopic to g.

Note that if we define $F : R^n \times 1 \to R^n \, by \, F(x,t) = (1-t^2)x,$ then F is also a homotopy from $f \; to \; g$. In other words, there can be several ways of deforming a map f into a given map g.

Example 2.2.3.This is a generalization of Example 3.2.2 above. Let X be any topological space and Y be a convex subset of $\mathrm{R}^n, i.e., Y$ has the property that whenever $y_1, y_2 \in Y,$ the line segment joining $y_1 to \, y_2$ is completely contained in Y. Let f,g: $X \to Y$ be any two continuous maps. Then f is homotopic to g. To see this, let us define the map $H : X \times I \to Y$ by

$$H(x,t) = tg(x) + (1-t) f(x).$$

Then we see at once that H is well-defined, continuous, it starts with f and terminates into g. A homotopy of this kind is called a straight-line homotopy.

Example 2.2.4. Let $S^1 = \{ z \in C : |z| = 1 \}$ be the unit circle. We know that we can also write $S^1 = \{ e^{i\theta} : 0 \le \theta \le 2\pi \}$. Define two maps $f, g : S^1 \to S^1 by \, f(z) = z \, and \, g(z) = -z, z \in S^1. Then \, f$ is homotopic to g and the map $F : S^1 \times I \to S^1$ defined by

$$F(e^{t\theta}, t) = e^{i\theta(\theta + t\pi)}$$

is a homotopy from $f \; to \; g$. Note that F is continuous because it is the composition of maps

$$S^1 \times I \to S^1 \times S^1 \to S^1$$
$$(e^{i\theta}, t) \to (e^{i\theta}, e^{it\pi}) \to e^{i(\theta + t\pi)},$$

Where the second map is multiplication of complex numbers. Note that in this example, the family of maps $\{f_t : S^1 \to S^1\}$ is just the family of rotations by the angle $t\pi, 0 \le t \le 1$.

The next result implies that the set of all maps from a space X to a space Y can be decomposed into disjoint equivalence classes.

Theorem 2.2.5.Let X, Y be fixed topological spaces and $C(X,Y)$ denote the set of all continuous maps from $X\ to Y$. Then the relation of "being homotopic to" is an equivalence relation in the set $C(X,Y)$.

Proof. Note that each continuous map $f : X \to Y$ is homotopic to itself because $H : X \times I \to Y$ defined by $H(x,t) = f(x)$ is a homotopy from f to itself. Next, suppose $H : f \cong g$. Then the map $H^i : X \times I \to Y$ defined by $H^i(x,t) = H(x,(1-t))$

is a homotopy from $g\ to\ f$; to see this, note that $H^i(x,0) = H(x,1) = g(x)$ and $H^{'}(x,1) = H(x,0) = f(x)\ for\ all\ x \in X$. Moreover, $H^{'}$ is continuous because $H^{'}$ is simply the composite of continuous maps.

$$X \times I \to X \times I \to Y,$$

Where the first map is the map $(x,t) \mapsto (x, 1-t)$ and the second H. The first map itself is continuous because its composite with the two projection maps, viz., $(x,t) \to x\ and\ (x,t) \to (1-t)$ is continuous. Thus, the relation is symmetric. Finally, suppose $H_1 : f \cong g\ and\ H_2 : g \cong h.$ Define a map $H : X \times I \to Y\ by$

$$H(x,t) = \begin{cases} H_1 & (x, 2t),\ 0 \le t \le 1/2, \\ H_2 & (x, 2t-1), 1/2 \le t \le 1. \end{cases}$$

Notes

Then H is continuous by the continuity lemma and $H(x,0) = f \; and \; H(x,1) = h, Hence \; H : f \cong h,$ proving the relation to be transitive.

The relation of homotopy in the set $C(X,Y)$ of all continuous maps, therefore, decomposes this set into mutually disjoint equivalent classes. The equivalence classes are called the homotopy classes of maps from $X \; to \, Y$ and the set of all homotopy classes is denoted by $[X,Y]$. We will need the following result very often.

Theorem 2.2.6. Let $f_1, g_1 : X \to Y$ be homeopathic and $f_2, g_2 : X \to Z$ be also homotopic. Then the composite maps $f_2 \circ f_1, g_2 \circ g_1 : X \to Z$ are homeopathic too, i.e., composites of homotopic maps are homotopic.

Proof: Let $H_1, f_1 \cong g_1 and \, H_2 : f_2 \cong g_2.$ Then, clearly, $f_2 \circ H_1 : X \times I \to Z$ is homotopy from $f_2 \circ f_1 to \; f_2 \circ g_1$. Next, define a map $H : X \times I \to Z$ by $H(x,t) = H_2(g_1(x),t), i.e.,$ the map H is simply the following composite:

$$X \times I \to Y \times I \xrightarrow{H_2} Z$$

$$(x,t) \to (g_1(x),t) \to H_2(g_1(x),t)$$

Then H is continuous and $H(x,0O = H_2(g_1(x),0) = f_2(g_1(x)), H(x,1) = H_2(g_1(x),1) = g_2(g_1(x)), i.e., H : f_2 \circ g_1 \cong g_2 \circ g_1,$

Now, because $f_2 \circ g_1 and \; f_2 \circ g_1$ is homotopic to $g_2 \circ g_1$, it follows by the transitive property of homotopy relation that $f_2 \circ f_1$ is homotopic to $g_2 \circ g_1$.

Exercise:

1. Let X be a topological space and $Y \subset S^2$ be the open upper hemisphere. Prove that any two maps $f, g : X \to Y$ are homotopic.

2. Let $P = \{p\}$ be a point space and X be a topological space. Show that X is path connected if and only if the set $[P, X]$ of homotopy classes of maps is a singleton.

3. Let X be a discrete space. Show that if a map $f : X \to X$ is homotopic to the identity map $I_X : X \to X$, then $f = I_X$. (Hint: The given condition implies that there is a path joining x and $f(x)$.)

4. Suppose X is a connected space and Y is a discrete space. Prove that the two maps $f, g : X \to Y$ are homotopic if and only if $f = g$.

5. Let S^1 be the unit circle of the complex plane and $f, g : S^1 \to S^2$ be two maps defined by $f(z) = z$ and $g(z) = z^2$. What is wrong in saying that the map $F : S^1 \times 1 \to S^1$ defined by $F(z, t) = z^{t+1}$ is a homotopy from $f \; to \; g$?

6. Let X be a locally compact Hausdorff space and the set $C(X, Y)$ of all continuous maps from $X \; to \; Y$ be given the compact open topology. Prove that two maps $f, g \in C(X, Y)$ are homotopic if and only if and only if these can be joined by a path in the space $C(X, Y)$.
(Hint: Use the exponential correspondence theorem.

2.3 CONTRACTIBLE SPACES AND HOMOTOPY TYPE

The notion of a contractible space is very important and the definition itself has some geometric appeal, as we shall see later. Recall that a map $f : X \to Y$ is said to be a constant map provided each point of X is mapped by f to some fixed point $\in Y$. If this is the case, then it is convenient to denote such a constant map by the symbol $C_{yo} i.e., C_{yo}(x) = y_0, \; for \; every \; x \in X$. We have

Definition 2.3.1. A topological space X is said to be a contractible space if the identity map $Ix : X \to X$ is homotopic to some constant map $C_x : X \to X$, where, of course, $x \in X$.

There are numerous examples of contractible spaces. For instance, we note that any convex subset of an Euclidean space R^n is contractible. For, let S be a convex subset of R^n. This means for any two points $x, y \in S$, the point $tx + (1-t)y$ *is also in* S *for all* $t, 0 \le t \le t$. Now let $x_0 \in S$. Define a map $H : S \times I \to S$ *by*

$$H : (x,t) = (1-t)x + tx_o$$

Then it is clear that H is a homotopy from the identity map on S to the constant map $C_{xo} : S \to S$. Hence, H is a contraction and so S is contractible. In particular, the Euclidean space R^n, the disk D^n are contractible spaces. More generally, a subspace X of R^n is said to be star-shaped if there exists a point $x_0 \in X$ such that the line segment joining any point of X to x_0 lies completely in X. For example, the subset $X \subset \mathrm{R}^{22}$(Fig.2.3) is star-shaped.

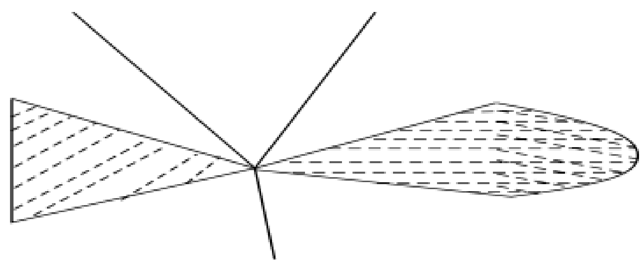

Fig.2.3: A star –shaped region

Then one can easily see that X is contractible to the point x_0 and the same contraction, as defined above, works in this case also. Now, we ask the following question: Determine whether or not the n-sphere S^n, $n \ge 1$ is contractible ? It is obviously not star-shaped. That does not mean, however, that X is not contractible (see Example 2.3.10). The answer to this question is “No”; it will take quite sometime before we can prove it. The following concept is again extremely basic.

Definition 2.4.2. Let $f : X \to Y$ be a continuous map. We say that f is a homotopy equivalence if there exists a continuous map $g : Y \to X$ such that that $g \circ f$ is homotopic to the identity map I_X *on* X *and* $f \circ g$ is homotopic to the identity map $I_Y \ on Y$. Two spaces $X \ and \ Y$ are said to be homotopically equivalent or of the same homotopy type of there exists a homotopy equivalence from one to the other.

We must observe that two homeomorphic spaces are of the same homotopy type. For, suppose $X \ and \ Y$ are homeomorphic and let $f : X \to Y$ be a homeomorphism. Then the inverse map $f^{-1} : Y \to X$ is continuous and satisfies the condition that $f^{-1} \circ f \cong I_X and \ f \circ f^{-1} \cong I_{Y.}$ This means f is homotopy equivalence, i.e., $X \ and \ Y$ are of the same homotopy type. The converse is not true. We have

Example 2.4.3. Consider the unit disk D^n (open or closed) and a point be the inclusion map and be the constant map. Then evidently $C_{X_o oi=IP.}$ On the other hand, the map and $C_{xo} : D^2 \to P$ be the constant map. Then evidently $C_{xo} oi = I_{P.}$ On the other hand, the map $H : D^2 \times I \to D^2$ defined by

$$H(x,t) = (1-t)x + tx_o$$

Is a homotopy from $I D^2 \ to \ i \circ C_{x0.}$ Thus, D^2 is of the same homotopy type as appoint space P and these are clearly not homoeomorphic.

One can easily verify that the relation of “homotopy equivalence” in the class of all topological spaces is an equivalence relation. Also, in view of the above example, the relation of homotopy equivalence is strictly weaker than the relation of “homeomorphism”. It is also clear from the above example that if a space X is compact then a space Y, which is homotopically equivalent to X, need not be compact, i.e., the compactness is not a homotopy invariant. Similarly, the topological dimension is not a homotopy invariant. Similarly, the topological dimension is not a homotopy invariant because dimension of the plane R^2 is 2 whereas a point has dimension zero. These and several other examples show that topological invaraiants are, in general, not “homotopy” invariants and so the homotopy classification of spaces is quite a weak classification .However, it is still very important because

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we will later define some homotopy invariants which would be evidently topological invariants. These invariants would also be computed for a large class of topological spaces. The moment we notice that if any of these invariants is not the "same" for any two given spaces $X \; and \, Y$, we can immediately assert that $X \; and \, Y$, are not of established strategy of algebraic topology for proving that two given spaces are not homoeomorphic. Example 3.4 is a trivial case of the following.

Theorem 2.4.4. A topological space X is contractible if and only if X is of the same homotopy type as a point space $P = \leq \{p\}$.

Proof. Suppose X is contractible. Let $H : X \times I \to X$ be a homotopy from the identify map I_X to the constant map $C_{x_o} : X \to X$. Define maps $i : P \to X \; and \; C : X \to P \, by \, i(p) = x_o and \; C(x) = p, x \in X$. Then, clearly, $c \circ i = I_P$. Also, the map

H is a homotopy from $J_X to \, i \circ C$ because $H(x,0) = x$ and

$H(x,1) = C_{x_o}(x) = x_o = i \circ C(x)$

For each $x \in X$. Hence $X \; and \; P$ are of the same homotopy type.

Conversely, suppose there are maps $f : X \to P \; and \; g : P \to X$ such that $g \circ f \cong I_X and \; f \circ g \cong I_{P.} \; Let \; g(p) = x_o and \; H : X \times I \to X$ be a homotopy from $I_X to \; g \circ f$. Then because $g \circ f(x) = g(f(x)) = g(p) = x_o, \; for \; all \; x \in X$. $g \circ f$ is the constant map $C_{xo} : X \to X$. Thus, I_X is homotopic to the constant map C_{xo} and so the space C_{xo} is contractible.

Thus, the contractible spaces are precisely those spaces which are homotopically equivalent to a point space. The intuitive picture of a contractible space. The intuitive picture of a contractible space X is quite interesting. A homotopy H which starts from the identity map on X and terminates into a constant map $C_{xo}, x_o \in X$, should be thought of as a continuous deformation of the space X which finally shrinks the whole space X into the point x_0. In other words, if we imagine the unit interval I as a time interval then at the time t=0, every point $x \in X$ is at its original place: as t-varies from 0 to 1 continuously, x moves continuously and approaches the point x_0; even x_0 moves accordingly and comes back to itself. Furthermore, all do not change abruptly. Thus, if we follow the movement of an arbitrary point $x \in X$, we note that it describes a path in X starting from x which terminates at x_0. In particular, we intuitively see that X is path connected. We make this statement precise:

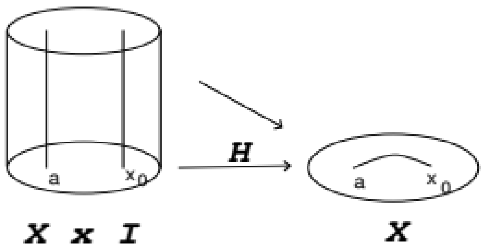

Fig. 2.3: Contractible space is path connected

Proposition 2.4.5.If X is a contractible space, then X is path connected.

Proof. Suppose is contractible to a point x_0 $and\ H : X \times I \to X$ is a homotopy from $I_X to C_{xo}$. Let $a \in X$. It suffices to show that a can be joined to x_o by a path in X .Note that H maps whole of bottom to itself whereas the entire top to the point x_0. Define a map $f : I \to X \times I\ by\ f(t) = (a,t)$ and note that it is continuous. Then $\omega(0) = G(f(0)) = H(a,0) = a,$

And

$\omega(1) = H(f(1)) = H(a,1) = x_0.$

Now suppose X is contractible. This means X can be contracted to some point $x \in X$. Can we then contract X to an arbitrary point $x_0 \in X$? The answer to this question is “yes”. We will now explain this. Let us prove

Propositon 2.4.6. A topological space X is contractible if and only if an arbitrary map $f:T \to X$ from any space $T\,to\,X$ is homotiopic to a constant map.

Proof :Suppose X is contractible. This means the identity map $I_X : X \to X$ is homotopic to some constant map. Say $C_{xo} : X \to X$. Now let $f:T \to X$ be any map. By theorem 2.2.6, we find that $I_X \circ f$ is homotopic to $C_{xo} \circ f$. But $I_X \circ f = f\; and\, C_{xo} \circ f : T \to X$ is the constant map.

For the converse, take $T = X$ and the map $f:T \to X$ to be the identity map. Then, by the given condition, we find that $I_X : X \to X$ is homotopic to a constant map, i.e., X is contractible.

It now follows from above that if X is a contractible space, then any map $f: X \to X$ is homotopic to a constant map $C_{xo} : X \to X$ In particular, for any $x \in X$, the constant map C_x and the identify map $I_X : X \to X$ both are homotopic to $C_{zo}, di.e., C_x$ is homotopic to I_x for all $x \in X$.

Corollary 2.4.7. If X is a contractible space, then the identity map $I_X : X \to X$ is homotopic to a constant map $C_x : X \to X$ for all $x \in X$. In particular, X can be contracted to any arbitrary point of X .

Once again, let X be a contractible space. We ask now a slightly stronger question. Can we contract X to some point $x_0 \in X$. so that the point x_0 does not move at all ? The answer to the question is “No”. We will give an example of the later on (See Exampl3 2.3.10). This question leads to the concept of relative homotopy which is stronger than the homotopy defined earlier

Definition 2.4.8.Let $A \subset X$ be an arbitrary subset and $f, g : X \to Y$ be two continuous maps. We will say that f is homotopic to g ”relative to A” if there exists a continuous map $F : X \to I \to Y$ such that

$$F(x,0) = f(x), F(x,1) = g(x),\ for\ all\ x \in X, and$$

$F(a,t) = f(a) = g(a), \, for \, all \, a \in A.$

Note that if we take A to be null set θ,then the concept of relative homotopy reduces to that of homotopy. It is also to be noted that if $f, g : X \to Y$ are to be homotopic relative to some subset $A of \; X, then \; f \; and \; g$ must agree on A to start with. The map f will change into the map g by a family continuous maps $h_t : X \to Y, t \in I,$ but the points of A will remain unchanged under h_t when t varies from 0 to 1.

If $C(X,Y)$ is a fixed subset and A denotes the set of all continuous maps $f : X \to Y,$ then following the proof of Theorem 2.2.5, one can prove that the relation of being "relativity homotopic to" with respect to A is an equivalence relation in the set F .

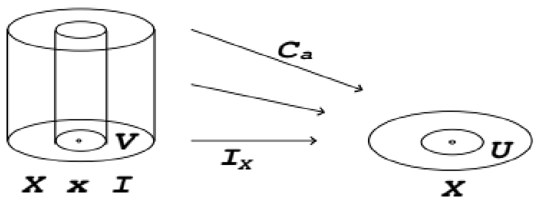

Fig. 2.5: Strongly contractible at a is semi-locally path connected at a.

Fig. 2.5: Strongly contractible to a point $A = \{a\}$ relative to the subset X , i.e., a can be contracted to the point a in point a , but the point a itself does not undego any change. In other words, we terminates into the constant map C_a relative to the subset $\{a\}$. This means, (see Fig. 2.5) under the continuous map F , the line X is mapped to the point $a \in X$. If we take any neighbourhood U of a , the continuity of F will give for each $t \in I,$ neighborhoods $V_t(a)$ of a in X and $W(t)$ of t in I such that $F(V_t(a) \times W(t)) \subset U.$ The compactness of I means that the open covering $\{W(t) : t \in I\}$ of I will have a finite sub cover, say, $W(t_1), ..., W(t_n),$ such that $F(V_{t_i}(a) \times W(t_i)) \subset U,$ for all $i = 1, ..., n.$ Therefore, $V(a) = \bigcap_{i=1}^{n} V_{t_i}(a)$ is a neighbourhood of a in X such that $F(V(a) \times I)$ in U , we find that b can be joined to a by a path which lies in U . This completes the proof of the following:

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Theorem 2.4.9.If a space X is contractible to a point $a \in X$ relative to the subset $\{a\}$, then for each neighbourhood U of a in X, there exists a neighbourhood V of a contained in U such that any point of V can be joined to a by a path lying completely inside U, i.e. X is semi-locally path connected.

Let us now consider the famous

Example 2.4.10. (Comb Space).We consider the following subset C (Fig.2.6) of the plane (shown only partly).

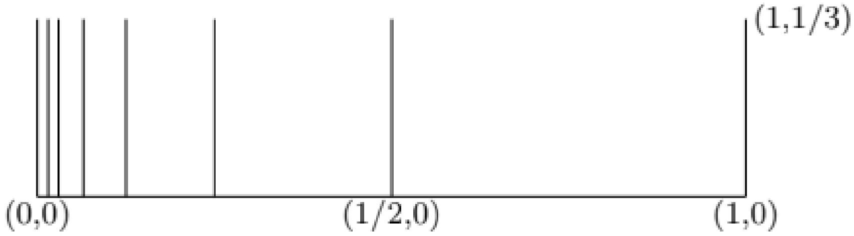

Fig. 2.6

It consists of the horizontal line segment joining (0.0) to (1,0) and vertical unit closed line segments standing on points (1/n,0) for each n=1,2,….together with the line segment joining (0,0) with (0,1).

The comb space C is contractible. The projection map $p : C \rightarrow L,$ where L is the line segment joining (0,0) with (1,0), is a homotopy equivalence. For, if i: $L \rightarrow C$ is the inclusion map, then $p \circ i = I_L$ and the map $F : C \times 1 \rightarrow C$ defined by $F((x, y), t) = (x, (1-t))y).$

Is a homotopy between $I_C and\, i \circ p.$ We already know that L is homeomorphic to the unit interval which is of the same homotopy type as a point space and so by Theorem 2.4.1, C is contractible.

C is not contractible relative to $\{(o,1)\}$ .Note that any small neighbourhood V of (0,1) has infinite number of path components. So, if we take the neighbour hood $U = D \cap C$ of (0,1)in C, where D is the open disk around (0,1) of radius 1./2, then U cannot have any neighbourhood V each of whose points can be joined to (0,1) by a path lying in U. This observation combined with Theorem 2.3.9, shows that C is not contractible relative to $\{(o,1)\}$.

Remark 2.4.11. Note that if $f,g:X\to Y$ are two continuous maps which are homotopic relative to some subset A of X, then obviously f is homotopic to g. The converse, however, is not true, i.e., there are maps $f,g:X\to Y$ which are homotopic, and even agree on a subset A of X, yet they need not be homotopic relative to A. For example, consider the identify map $I_X : X\to X$ of the comb space $X=C$ and the constant map $C_{(0,1)} : X\to X$ Then, obviously, I_X and $C_{(0,1)}$ are not homotopic relative to (0,1). This means the concept of relative homotopy is definitely stronger than that of homotopy.

There are some fundamental concepts related to "contractible" spaces. It is appropriate to discuss them no.

Definition 2.4.12. Let $A\subset X$. We say that A is a retract of X if there exists a continuous map $r:X\to A$ such that $r(a)=a,\ \mathit{for\ all}\ a\in A$. The map r is called the retraction map.

If $i:A\to X$ is the inclusion map, then the condition $r(a)=a$, for all $a\in A$ is equivalent to saying that $r\circ i=I_A$. Thus, A is a retract of X if and only if the inclusion map I has left inverse. As an example, note that every single point $x_0\in X$ of an arbitrary topological space $\{(0,1)\}$ is a retract of X and the constant map C_{x_o} is the retraction map. Let X = [0,1], the unit interval and $A=\{0,1\}$, the boundary of X. Then A cannot be a retract of X because X is connected whereas A is disconnected. Let us ask a general question: Let X be the retract of X. The answer is again "No" and this fact is known as $Brouwer's$ No Retraction theorem. The proof, however, will be given later on.

Defination 2.4.12.A topological space X is said to be deformable into a subspace $A\subset X$ if there is a map $f:X\to A$ which is right homotopy inverse of the inclusion map $i:A\to X$, *i.e.*, the identity map I_X is homotopic X.

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The above definition asserts that there is a homotopy, say, $D: X \times I \to X$ such that $D(x,0) = x, D(x,1) = i(f(x)) = f(x)$.

Any such homotopy D is called a deformation of X into A and we say that X is deformable into A. It must be observed that the homotopy D ,which starts with identity map $I_X : X \to X$, simply moves each point of X continuously, including the points of A and finally pushes every point of X into a point of A. In particular, if a space X is deformable into a point $a \in X$, then X is contractible and vice versa. If we can find a deformation D which deforms X into A but the points of A do not move at all, then the homotopy D will be "relative to A" and we say that X is strongly deformable into A. In such a case, note that for each $a \in A, D(a,1) = f(a) = a$ and so the map $f : X \to A$ is automatically a retraction of X onto A. We have

Definition 2.4.14. A space X is said to be strongly deformable into a subspace A if there is a continuous map $f : X \to A$ which is the right homotopy inverse of the inclusion map $i : A \to X$ relative to A, i.e., the identity map $I_X : X \to X$ is homotopic to $i\, o f : X \to X$ relative to A.

Clearly, if X is strongly deformable into a subspace A, then it is also deformable into A. But the converse is not true even if the map $f : X \to A$ is onto: the comb space C (Example 2.3.10) is deformable into the point $\{(0,1)\}$ because it is contractible and so the identity map I_X is homotopic to the map $i \circ C_{(o,1)}$ where $C_{(0,1)}$ is the constant map $X \to X$. However, we have already seen that I_X is not homotopic to $i \circ C_{(o,1)}$ relative to the point $\{(0,1)\}$. Finally, we have

Definition 2.4.15 A subspace a of a topological space X is said to be a deformation retract of X of X is deformable into A so that the final map is a retraction of X and A.

Definition 2.4.16 A subspace A of a topological space X is said to be a strong deformation retract of X if X is deformable into A strongly(A is then automatically a retract of X).

It follows from the above definitions that if A is a deformation retract of X, then the inclusion map $i: A \to X$ has a two-side homotopy inverse, i.e., it is a homotopy equivalence, and consequently A and X are of the same homotopy type . On the other hand, if A is a strong deformation retract of X, then A is homotopic ally equivalence, and consequently A and X are of the same homotopy type. On the other hand, if A is a strong deformation retract of X, then A is homotopically equivalent to X and something more is true, viz., X can be deformed into A without moving the points of A at all. For instance, the point (0,1) of the comb space X is a deformation retract of X but is not a strong deformation retract of X. Quite often we will use the following.

Example 2.4.17.For $n \geq 1, S^n \subset R^{n+1} - \{(o,...,0)\} = X$ is a strong deformation retract of X .fig. 2.7 depicts the case n=1.

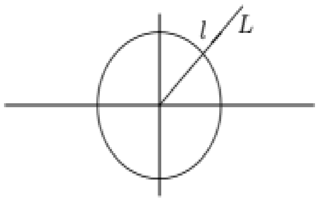

Fig. 2.7

Each infinite line segment L starting from its origin intersects then circle S^1 at exactly one point, say, L.Since origin is not a point of $R^2 - \{(0,0)\}$, these lines are disjoint and their union is $R^2 - \{(0,0)\}$. We define a map $r: R^2 - \{(0,0)\} \to S^1$ by $r:(x) = 1$ for all points $x \in L$. Then, clearly r is continuous and S^1 becomes a retract of $R^2 - \{(0,0)\}$. Let us define a deformation $D:(R^2 - \{(0,0)\} \times I \to R^2 - \{(0,0)\}$ by

Contractible Spaces and Homotopy Type

$$D(x,t) = (1-t)x + t\frac{x}{\|x\|}.$$

Then D is clearly a strong deformation retraction of $R^2 - \{(0,0)\}$ rrelative to S^1 int o S^1.ıA similar argument shows that S^n ıis a strong deformation retract of $R^{n+1} - \{(0,0)\}$.

Exercise:

1. Let $A \subset X$ be a retract of X where X is Hausdorff. Then prove that A must be closed in X. (This implies that an open interval (0,1) can never be a retract of any closed subset of the real line.)
2. Let X be a connected space and $x_{0,} x_1 \in X$ be two points of X which have disjoint open neighbourhoods in X. Show that $A = \{x_0, x_1\}$ can never be a retract of X.
3. Prove that a space X is contractile if and only if every map $f : X \to T$ to any space T is null-homotopic.
4. Show that of A is a strong deformation retract of X and B is a strong deformation retract of A, then B is a strong deformation retract of X
5. Prove that an arbitary product of contractible space is again contractible.
6. Prove that a retractof a contractible space is contractible.
7. For any space X consider the cylinder $X \times I$ over X and collapse the top $X \times I$ of this cylinder to a point. The resulting quotient space, called cone over X, is denoted by $C(X)$. Prove that $C(X)$ is contractile for any space X.
8. Let $I^2 = [0,1] \times [0,1]$ be the unit square and $C \subset I^2$ be the cob space (Example 2.3.10). Prove that C is not a retract of I^2. (Hint: Given any open neighbourhood $U = B((0,\frac{1}{2}),\frac{1}{4}) \cap C$,there exist a connected neighbourhood $V \subset U$ of $(0,\frac{1}{2})$ in $I \times I$ such that $r(V) \subset U$,but $r(V)$ is disconnected.)
9. Determine which of the following spaces are contractible:

(i) Unit interval I=[0.1].

(ii) $S^2 - \{p\}, where\, S^2$ is a 2-sphere and p is any point of S^2.

(iii) Any solid or hollow cone in R^3

(iv) The subspace $\{0\} \cup \{1/n :\in N\}$ of real line.

10. Consider the following subspace X of plane R^2, X consist of all closed line segments joining origin with points $\left(1, \frac{1}{n}\right), n \geq 1$ and the line $\{(x,0) | 0 \leq x \leq 1\}$. Prove that X is contractile, but none of the points $(x,0), x \geq 0,$, is a strong deformation retractof X.

11. Let I be the unit interval and X be any path connected space. Prove that the sets [I,X]and [X,I] each has only one element. (Hint: The space I is contractible)

12. Give an example of a space X which is of the same homotopy type as a discrete space $D = \{0,1,2,3\}$, , but is not homeomorphic to D.

13. Prove that a homotopy invariant is also topological invariant. Give an example to show that a topological invariant need not be a homotopy invariant. (Hint: There are contractible spaces which are not compact, not locally compact, not locally connected etc.)

Check Your progress

1. Prove: Theorem: Let be fixed topological spaces and denote the set of all continuous maps from Then the relation of "being homotopic to" is an equivalence relation in the set .

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2. Prove: Theorem: Let be homeopathic and be also homotopic. Then the composite maps are homeopathic too, i.e., composites of homotopic maps are homotopic.

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3. Prove: A topological space is contractible if and only if is of the same homotopy type as a point space

2.4 LET US SUM UP

Let X be a topological space. Often we associate with X an object that depends on X as well as on a point x of X. The point x is called a base point and the pair (X, x) is called pointed space.

Each pointed space (X, x), we can associate a group (X, x), called the fundamental group of the space X at x.

Let X, Y be two spaces and $f, g : X \to Y$ be two continuous maps. We say that f is homotopic to g $(\textit{and denote it by writing } f \cong g)$ if there exists a continuous map $F : X \times I \to Y$ such that $F(x,0) - f(x)$ and $F(x,1) = g(x)$ *for all* $x \in X$. The map F is called a homotopy from f *to* g.

Let X, Y be fixed topological spaces and $C(X,Y)$ denote the set of all continuous maps from X *to* Y. Then the relation of "being homotopic to" is an equivalence relation in the set $C(X,Y)$.

Let $f_1, g_1 : X \to Y$ be homeopathic and $f_2, g_2 : X \to Z$ be also homotopic. Then the composite maps $f_2 \circ f_1, g_2 \circ g_1 : X \to Z$ are homeopathic too, i.e., composites of homotopic maps are homotopic.

A topological space X is contractible if and only if X is of the same homotopy type as a point space $P = \leq \{p\}$.

2.5 KEY WORDS

Topological space

Fundamental group

Isomorphism

Homeomorphic

Homotopy

homotopy equivalence

2.6 QUESTIONS FOR REVIEW

1. Explain about Homotopy
2. Explain about contractible spaces and homotopy
3. Determine which of the following spaces are contractible:

(i) Unit interval I=[0.1].

(ii) $S^2 - \{p\}, where\, S^2$ is a 2-sphere and p is any point of S^2.

(iii) Any solid or hollow cone in R^3

(iv) The subspace $\{0\} \cup \{1/n :\in N\}$ of real line.

2.7 SUGGESTIVE READINGS AND REFERENCES

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2.8 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. See section 2.2.5
2. See section 2.2.6
3. See section 2.4.4

UNIT-3 PROPERTIES OF FUNDAMENTAL GROUP

STRUCTURE

3.0 OBJECTIVE

In this unit we will learn about and understand about product of paths in topological spaces, properties of fundamental group, important definitions, theorems and propositions.

3.1 INTRODUCTION

In the mathematical field of algebraic topology, the fundamental group of a topological space is the group of the equivalence classes under homotopy of the loops contained in the space. It records information about the basic shape, or holes, of the topological space. The fundamental group is the first and simplest homotopy group. The fundamental group is a homotopy invariant—topological spaces that

Notes

are homotopy equivalent (or the stronger case of homeomorphic) have isomorphic fundamental groups.

The abelianization of the fundamental group can be identified with the first homology group of the space. When the topological space is homeomorphic to a simplicial complex, its fundamental group can be described explicitly in terms of generators and relations.

3.2 PRODUCT OF TWO PATHS IN TOPOLOGICAL SPACES

Recall that a path in a topological space X is just a continuous map $\alpha : I = [0,1] \to X ; \alpha(0)$ is called the Initial point and $\alpha(1)$ is called the terminal point of the path $\alpha, If\ \alpha, \beta$ are two paths in X such that $\alpha(1) = \beta(0),$ then we can define a new path (Fig.2.8), called the product of $\alpha\ and\ \beta$ denoted by $\alpha * \beta$, as follows:

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & 0 \le t \le 1/2 \\ \beta(2t-1) & 1/2 \le t \le 1 \end{cases} \qquad (3.1)$$

Note that $\alpha * \beta : 1 \to X$ is continuous by the continuity lemma, the initial point of $\alpha * \beta$ is the initial point of α and the terminal point of $\alpha * \beta$ is the terminal point of β.

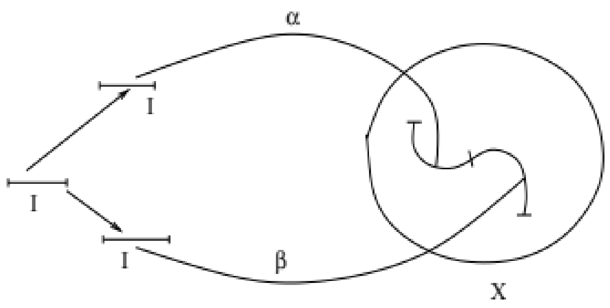

Fig.: Product of two paths

One can verify that if α, β, γ are three paths in X such that $\alpha(1) = \beta(0), \beta(1) = \gamma(0),$ then $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma)$ are paths in X which are not necessarily the same paths. To see this, just apply definition (3.4.1) and look at the image of the point t=1/2 under both the paths. Therefore, the product of paths is not an associative operation. Let us now fix a point $x_0 \in X$ and consider the set of all closed paths at x_0, i.e., those paths whose initial and terminal points are x_0; such a path is also known as loop in X based at x_0. It is clear that the product of two loops based at x_0 is always defined. The difficulty that the product of loops based at x_0 need not be associative is still there. To surmount this difficulty and to finally get a group structure, we will introduce an equivalence relation in the set of all loops in X based at $x_0 \in X$. First, we have an important as well as general

Definition 3.4.1. Let α, β be two paths in X with the same initial and terminal points, i.e., $\alpha(0) = \beta(0) = x_0, \alpha(1) = \beta(1) = x_1$. We will say that α is equivalent to β, and write it as $\alpha R_{(x_0,x_1)} \beta$, if there exists a homotopy between $\alpha \, and \, \beta$ relative to the subset $\{0,1\} \, of \, I$. of I. In other words, the homotopy keeps the end points fixed.

Thus, the path α is equivalent to β if there exists a continuous map $H : I \times I \to X$ such that

$$H(s,0) = \alpha(s), H(s,1) = \beta(s)$$

$$H(0,t) = \alpha(0) = \beta(0), H(1,t) = \alpha(1) = \beta(1)$$

For all $s \in I$ and for all $t \in I$. In other words, the path α changes continuously and finally it becomes the path β, but during all this transformation the end points remain fixed (Fig. 3.9).

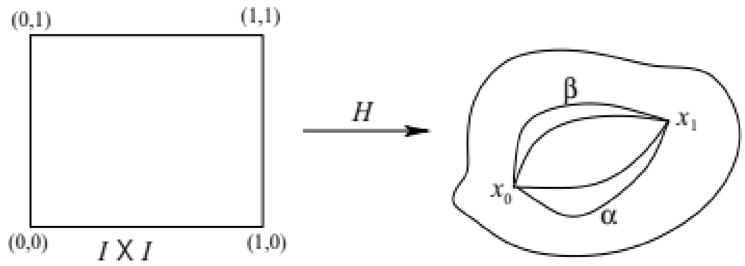

Fig. 3.9: H is a path-homotopy

Notes

The relative homotopy H , which keeps the end points fixed, is sometimes called path homotopy, just for convenience. The next result, which is in fact, true, more generally, or homotopy relative to any subset A of the domain space (of. Definition 2.3.8), show that the above relation is an equivalence relation.

Therorem 3.4.2. Let $x_0, x_1 \in X$. Then the relation of "being equivalent" in the set of all paths starting from x_0 and terminating at x_1 is an equivalence relation.

Proof. Let $\alpha : I \to X$ be any path with $\alpha(0) = x_0, \alpha(1) = x_1$. Then the map $H : I \times I \to X$ defined by $H : (s,t) = \alpha(s)$ is a homotopy from α to itself relative to $\{0,1)\}$. Thus, the relation is reflexive. Next, suppose α, β are two paths from $x_0 to\, x_1 and H : I \times I \to X$ is a continuous map such that

$$H(s,0) = \alpha(s), H(s,1) = \beta(s), \forall_s \in I$$

$$H(0,t) = x_0, H(1,t) = x_1, \forall t \in I$$

Define a map $H^{'} : I \times I \to X\ by$

$H^{'}(s,t) = H(s, I - t).$

Then H' is continuous and has the property that

$H'(s,0) = H(s,1) = \beta(s), s \in I$

$H'(0,t) = x_0 H'(1,t) = x_1, \forall t \in I$.

Thus, β relative to $\{0,1\}$ implies relative to $\{0,1\}$ i.e., the relation is symmetric. To prove the transitivity of the relation, suppose X relative to $\{0,1\}$, X relative to $\{0,1\}$. Let H_1 and H_2 be two homotopies such that

$$H_1(s,0) = a(s), H_1(s,1) = \beta(s), \forall s \in I$$

$$H_1(0,t) = x_o H_1(1,t) = x_1, \forall t \in I$$

And

$$H_2(s,0)=\beta(s), H_2(s,1)=\gamma(s), \forall s\in I$$

$$H_2(0,t)=x_o, H_2(1.t)=x_1, \forall t\in I.$$

Define a map $H: I\times I\to X$ by

$$H(s,t)=\begin{cases} H_1(s,2t) & 0\le t\le 1/2 \\ H_2(s,2t-1) & 1/2\le t\le 1 \end{cases}$$

Then H is continuous by the continuity lemma and is indeed a homotopy relative to $\{0,1\}$ from α to γ. This completes the proof of the theorem.

When $x_0=x_1,$ we conclude the following result.

Corollary 3.4.3.The relation "being equivalent' in the set of all loops in X based at $x_0\in X$ is an equivalence relation.

Next, we are go into to deal with only loops in X based at given point $x_0\in X$. If α,β are two loops based at X which are equivalent, then we will write this as $R_{x_0}\beta$ .Also, the equivalence, class of a loop α based at x_0 will be denoted by the symbol $[\alpha]$ and called the homotopy class of the loop α. It must be emphasized at this point that if α is treated as a map $\alpha: I\to X$ with $\alpha(0)=\alpha(1)$, then the homotopy class of the map α, according to Theorem 3.2.5 is different from the path homotopy class of loop α specified by Theorem 3.4.2. In fact, the former homotopy class is, in general ,larger than the latte path, homotopy class. Let $\pi_1(X,x_0)$ denote the set of all homotopy classes of loops in X based at x_0, i.e.,

$$\pi_1(X,x_0)=\{:[\alpha] is\, a\, loop\, X\, based\, at\, x_0\}$$

The next proposition implies that the product of loops induces a product in the set of al homotopy classes of loops based at x_0. Recall that if

a, β are two loops at x_0 ,then their product $\alpha * \beta$ is also a loop at $x_0 \in X$.

Proposition 3.4.4. Suppose $\alpha, \beta, \alpha^{'} . \beta^{'}$ are loops in X based at

$If\ \alpha R_{xo} \alpha^{'}, \beta^{'} R_{x_0} 3^{'} R x_0\ then\ \alpha * \beta R_{x_0} R_{x_o}$

Proof. Let X_1 be a homotopy from α', H^2 be a homotopy from $\beta to \beta^{'}$

i.e., H_1, H_2 are maps from $I \times I \to X$

3.3 FUNDAMENTAL GROUP AND ITS PROPERTIES

$$H_1(s,o) = \alpha(s), H_1(s,1) = \alpha^{'}(s), \forall_s \in I$$

$$H_2(o,t) = x_0 = H_1(1,t), \forall t \in I$$

and

$$H_2(s,0) = \beta(s), H_2(s,1) = \beta^{'}(s), \forall s \in I$$

$$H_2(0,t) = x_0 = H_2(1,t), \forall t \in I.$$

Define a map $H : I \times I \to X\ by$

$$H(s,t) = \begin{cases} H_1(2s,t) & 0 \le s \le 1/2 \\ H_2(2s-1,t) & 1/2 \le s \le 1 \end{cases}$$

Then H is continuous by continuity lemma, and

$$H(s,0) = \begin{cases} H_1(2s,o) & 0 \le s \le 1/2 \\ H_2(2s-1,0) & 1/2 \le s \le 1 \end{cases}$$

$$= \begin{cases} \alpha(2s) & 0 \le s \le 1/2 \\ \beta(2s-1) & 1/2 \le s \le 1 \end{cases}$$

$$= (\alpha * \beta)(s).$$

By a similar calculation,

$H(s,1) = (\alpha^{'} * \beta^{'})(s)$ *and* $H(0,t) = x_0 = H(1,t), \forall t \in 1.$*Thus* $\alpha * \beta \simeq_{x_0} \alpha^{'} * \beta^{'}.$

One can easily observe that the above proof yields the following general result, viz.,

Corollary 3.4.5. Let $\alpha^{'}$ be two path homotopic paths joining x_0 with x_1 and $\beta, \beta^{'}$ be two path homotopic paths joining $x_2 to\, x_2$. Then $\alpha * \beta$ is path-homotopic to $\alpha * \beta$ joining $x_0 to x_0$ and the path homotopy can be chosen so that the point x_1 remains fired.

Sometimes we will need to consider the path homotopy classes of paths joining $x_0 to\, x_1$. If α is a path joining $x_0 with\, x_1$ then $[\alpha]$ will also be used to denote the path homotopy class represented by α. The set of all path homotopy classes of paths joining $x_0 to\, x_1$. will be denoted by $\pi_1(X, x_0\, x_1)$. In this terminology, we can define an operation

$$\circ : \pi_1(X, x_0\, x_1) \times r_1(x, x_1, x_2) \rightarrow \pi_1(X, x_0, x_2)$$

By

$$[\alpha] \circ [\beta] = [\alpha * \beta]$$

The above corollary says that the map $\circ$ is well defined. In case x_0, x_1, x_2 are the same points, the map defines a binary operation in the set $\pi_1(X, x_0)$ of all homotopy classes of loops based at x_0. Consequently, we can state the next

Definition 3.4.6. Let $[\alpha], [\beta]$ be any two elements of $\pi_1(X, x_0)$. Then we define their product $[\alpha] \circ [\beta]$

$$[\alpha] \circ [\beta] = [\alpha * \beta].$$

The following basic result can now be proved.

Theorem 3.4.7. The set $\pi_1(X, x_0)$ of all path homotopy classes of loops based at x_0 is a group with respect to the binary operation "$\circ$" define above.

Proof. We must prove that the operation $\circ$ is associative, there exists identity element in $\pi_1(X, x_0)$ and each element of $\pi_1(X, x_0)$ has an inverse in $\pi_1(X, x_0)$. The proof of each of these statements is achieved by constructing a suitable path homotopy relative to $\{0,1\}$ between appropriate paths and we discuss them below one by one:

The operation is associative: Let $[\alpha],[\beta],[\gamma]$ be three elements of $\pi_1(X, x_0)$. Since

$$([\alpha]\circ[\beta]\circ[\gamma] = [(\alpha*\beta)*\gamma]$$

And

$$([\alpha]\circ[\beta]\circ[\gamma] = [\alpha*(\beta*\gamma)],$$

It is sufficient to show that $(\alpha*\beta)*\gamma R_{x_0} \alpha*(\beta*\gamma)$. By definition

$$((\alpha*\beta)*\gamma(s) = \begin{cases} \alpha(4s) & 0 \le s \le 1/4 \\ \beta(4s-1) & 1/4 \le s \le 1/2 \\ \gamma(2s-1) & 1/2 \le s \le 1 \end{cases}$$

And

$$(\alpha*(\beta*\gamma))(s) = \begin{cases} \alpha(2s) & 0 \le s \le 1/2 \\ \beta(4s-2) & 1/4 \le s \le 3/4 \\ \gamma(4s-3) & 3/4 \le s \le 1. \end{cases}$$

Thus we should define a homotopy $H: I \times I \to X$ such that

$$\begin{cases} X(s,0) = ((\alpha*\beta)*\gamma)(s), H(s,1) = (\alpha*\beta*\gamma)(s) \\ H(0,t) = x_0 = H(1,t), \forall s,t \in I \end{cases}$$

(3.6.1)

Such a homotopy is given by the formula

$$H(s,t)=\begin{cases}\alpha(4s/(t+1)) & 0\le s\le (t+1)/4\\ \beta(4s-1-t) & (t+1)/4\le s\le (t+2)/4\\ \gamma((4s-2-t)/(2-t)) & (t+2)/4\le * \le 1\end{cases}$$

The motivation for writing this homotopy comes from Fig.3.10.

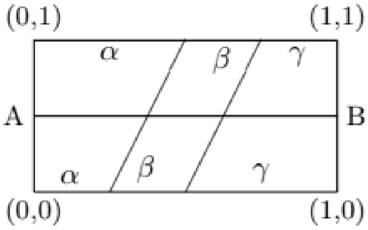

Fig. 2.10

We divide the square $I\times I$ into three quadrilaterals:

$Q_1 : 0\le s\le (t+1)/4$
$Q_2 : (t+1)/4\le s\le (t+2)/4$
$Q_3 : (t+2)/4\le s\le 1, t\in I$

For instance, the equation of the line joining (1/4,0) and (1/2,1) would be $t=4s-1$ and so the equation of the region Q_1 would be

$$0\le s\le (t+1)/4, 0\le t\le 1.$$

For a fixed $t\in I$, a typical horizontal line AB would have three pars. When t moves from 0to1, these three parts also change their positions. For t=0, we get a partition defining $(\alpha*\beta)*\gamma$ and for $t=1$, we get a partition define $\alpha*(\beta*\gamma)$. The map H is defined by $\alpha\, on Q_1, \beta\, on Q_2 \gamma\, on Q_{3,}$ each of which is continuous. On their common boundary, the two definitions match yielding a nice map H. Hence, by the continuity lemma. H is continuous. Moreover, conditions (3.6.1) are evidently satisfied. This completes the proof that X is associative.

Remark 3.4.8.If α is a path joining x_0 with x_1, β is a path joining x_1 with x_2 and γ is a path joining x_2 with x_3, then the same proof as above says, more generally ,that

$$([\alpha]\circ[\beta]\circ[\gamma]=\left[(\alpha*\beta)*\gamma\right]$$

$$=[\alpha*(\beta*\gamma)]$$

$$=[\alpha]\circ([\beta]\circ[\gamma]).$$

There exits an identify element in $\pi_1(X,x_0)$: Let us consider the constant loop $C_{x0}: I \to X$. We claim that the class $[C_{xo}] \in \pi_2(X,x_0)$ is an identity element ,i.e., for each loop $\alpha\ in\ X$ based at x_0, we show that $C_{x_0}*\alpha R_{x_0}\alpha$ and $\alpha * C_{xo} * R_{xo}\alpha.$ For this let us consider Fig.2.11.

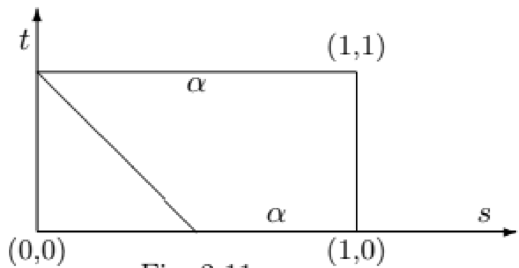

Fig. 2.11

Define a map $H: I\times I \to X\ by$

$$H(s,t)=\begin{cases} x_0 & 0\le s\le(1-t)/2 \\ \alpha(2s+t-1)/(t+1)) & (1-t)/2\le s\le 1\end{cases}$$

Note that the equation of the line joining (1/2,0) and (0,1) is X .H maps the whole triangle below this line to the point X and the two definitions of H on the line X match. The continuity of H follows from the continuity lemma. Furthermore,

$$H(s,0)=\begin{cases} x_0 & 0\le s\le 1/2 \\ 0(2s-1) & 1/2\le s\le 1\end{cases}$$

$$=(C_{x0}*\alpha)(s)$$

$$H(s,1)=\alpha(s).$$

and $H(0t)=x_0=H(1,t)$. This shows that $C_{x0}*\alpha R_{x0}\alpha$. To prove that $[C_{x0}]$ is also right identity, we can write a suitable path homotopy by looking at Fig. 3.12.

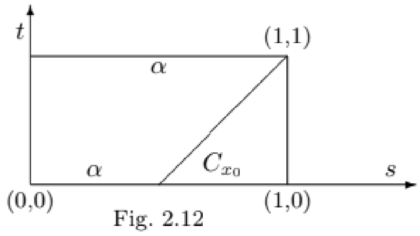

Fig. 2.12

This would be

$$H(s,t)=\begin{cases}\alpha(2s/1+t)) & 0\le s\le (t+1)/2\\ x_0 & (1+t)/2\le s\le 1\end{cases}$$

More generally, we have

Remark 3.4.9. Let α be any path joint x_0 with x_1. Then the above proof implies

(1)$[C_{xo}]\circ[\alpha]=[C_{xo}*\alpha]=[\alpha],$

(2)$[\alpha]\circ[C_{x1}]\circ=[\alpha*C_{x1}]=[\alpha].$

In other words, C_{xo} serves as the left identity and C_{x1} serves as the right identity for any $[\alpha]$

Each element of $\pi_1(X,x_0)$ has an inverse: let $[\alpha]\in\pi_1(X,x_0).$ We choose a representative, say α, of the homotopy class $[\alpha].$ For this α ,we define a loop $\alpha^{'}:I\to X$ by

$$\alpha^{'}(t)=\alpha(1-t).$$

Geometrically, $\alpha^{'}$ simply describes the same path as α, but in reverse direction, .We claim that the homotopy class $[\alpha^{'}]\in\pi_1(X,x_0)$ does not depend on the choice of α from the class $[\alpha].$ For, suppose $\beta Rx_0\,\alpha$ by a homotopy H. Then we can define a homotopy $H^{'}:I\times I\to X$ by

$$H^{'}(s,t)=H(1=s,t).$$

Then, obviously, $X^{'}$ is a homotopy from $\beta^{'} to\, \alpha^{'}, i.e.,[\beta^{'}]=[\alpha^{'}].$ .Now we claim that

$$[\alpha^{'}]\circ[\alpha]=[C_{xo}]=[\alpha]\circ[\alpha^{'}].$$

For this, we must construct a homotopy from X relative to X We now consider Fig. 3.13.

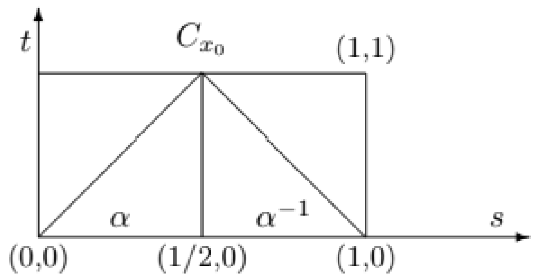

Fig.3.13

Just as in earlier cases, the required homotopy $H: I\times I\to X$ is given by

$$H(s,t)=\begin{cases} x_0 & 0\le s\le t/2 \\ \alpha(2s-t) & t/2\le s\le 1/2 \\ \alpha(2-2s-t) & 1/2\le s\le 1-t/2 \\ x_0 & 1-t/2\le s\le 1. \end{cases}$$

It is easily verified that H is well defined, continuous and has the required properties. By exactly a similar argument we can show that

$$[\alpha^{'}]\circ[\alpha^{-1}]=[C_{x0}].$$

Thus , $[\alpha']$ is an inverse of $[\alpha'] in\, \pi_1(X,x_0)$.

Remark 3.4.10.The same proof says that, more generally, if α is a path joining $x_0 with\, x_1$, then its inverse path α^{-1} has the following properties.

$$[\alpha]\circ[\alpha^{-1}]=[C_{xo}],$$
$$[\alpha^{-1}]\circ[\alpha]=[C_{x1}].$$

Remark 3.4.11. In the proof of the preceding theorem, it was enough to show that $\pi_1(X,x_0)$ has a left identify and each element of $\pi_1(X,x_0)$ has a left inverse -this is a result from elementary group theory. However, we have shown the existence of two-sided identity and two-sided inverse only to give more practice of writing path homotopies.

Remark 3.4.12. We also note that if $\alpha, \beta, \gamma, \delta$ are four loops based at $x_o \in X$, then by the associative law proved above, we find that

$$(\alpha * \beta) * (\gamma * \delta) R_{x0} \alpha * (\beta * (\gamma * \delta)) R_{x0} ((\alpha * \beta) * \gamma) * \delta$$

and so the generalized associative law is valid in the sense that placing of parentheses does not make any difference in the homotopy class. Therefore, we can just ignore the parentheses and write the above loop only as $[\alpha * \beta * \gamma * \delta]$.

Definition 3.4.13. Let X be a topological space and $x_0 \in X$. Then the group $\pi_1(X, x_0)$ obtained in Theorem 2.4.7 is called the fundamental group or the Poincare group of the space X based at $x_0 \in X$.

Remark 3.4.14. At this stage one would like to see examples of fundamental group so some spaces. We will give several examples somewhat later, but before that it would be helpful to prove a few results on the behaviour of fundamental group so that one can have some idea about the possibilities of the nature of fundamental group of a given space. We should also point out here that it is in the very nature of algebraic topology that computing associated algebraic objects is normally a long process and sometime can also be extremely difficult.

Having defined the fundamental group of a space X based at a point $x_0 \in X$, one would naturally like to ask: how important is the role of base point x_0 in the group $\pi_1(X, x_0)$? How are $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ related if $x_0 \neq x_1$? In fact, if X is arbitrary, then a loop at x_0 being itself path connected, will be completely in the path component of x_0 and so if x_0, x_1 are in distinct path components of X, then $\pi_1(X, x_o), \pi_1(X, x_1)$ are not related at all. However, if x_0, x_1 belong to the same path component of X then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are indeed isomorphic. This follows from the next

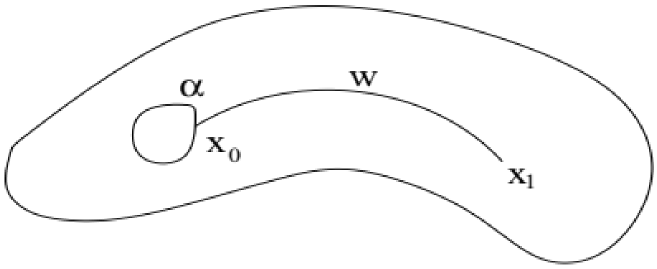

Fig.3.14

Theorem 3.4.15. Let X be a path connected space and x_0, x_1 be any two points of X . Then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic. In fact, each path joining $x_0 to x_1$ defines on isomorphism from $\pi_1(X, x_0) to \pi_1(X, x_1)$.

Proof. Consider Fig.3.14 Let $\omega I \to X$ be a path joining $x_o to x_1$ and suppose ω^{-1} is the inverse path of $\omega, i.e., \omega^{-1}(t)\omega(1-t)$ for each $t \in I$. If a X is any loop based at x_o, then it is clear that $\omega^{-1} * \alpha * \omega$ is a loop based at x_1. Thus, we can define a map

$$P_\omega : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

by

$$P_\omega : [\alpha] = [\omega^{-1} * \alpha * \omega].$$

First, we check path P_ω is well define, Suppose $\alpha \square_{xo} \beta$ and let $H : I \times I \to X$ be a homotopy relative to $\{0,1\}$ from $\alpha to \beta$ .Then by Corollary 2.4.2, $\omega^{-1} * \alpha * \omega^{-1} * \beta * \omega,$ which means $P_\omega[\alpha] = P_\omega[\beta].$ Now, let us see the following computation:

$$P_\omega([\alpha] \circ [\beta]) = P_\omega[\alpha * \beta]$$

$$= [\omega^{-1} * (\alpha * \beta) * \omega]$$

$$= [\omega^{-1} * \alpha] \circ [\beta * \omega]$$

$$= [\omega^{-1} * \alpha * C_{x0}] \circ [\beta * \omega]$$

$$= [\omega^{-1} * \alpha * \omega] \circ [\omega^{-1} * \beta * \omega]$$

$$P_{\omega}[\alpha] \circ P_{\omega}[\beta].$$

If we use ω^{-1} instead of ω, then we get a homomorphism

$$P_{\omega^{-1}} : \pi_1(X, x_1) \to \pi_1(X, x_0.)$$

Now, for each $[\alpha] \in \pi_1(X, x_0)$, we have

$$P_{\omega^{-1}} \circ P_{\omega}[\alpha] = P_{\omega^{-1}}[\omega^{-1} * \alpha * \omega]$$

$$= [\omega * \omega^{-1} * \alpha * \omega * \omega^{-1}]$$

$$= [\alpha],$$

Which means $P_{\omega-1} \circ P_{\omega}$ is identity on $\pi_1(X, x_0)$. By a similar argument, we see that $P_{\omega} \circ P_{\omega^{-1}}$ is identity on $\pi_1(X, \pi_1)$. It follows that P_{ω} is an isomorphism.

Remark 3.4.16. It follows from the above theorem that for a path connected space X, the fundamental group $\pi_1(X, x)$ is independent of the base point x up to isomorphism of groups. Therefore, for a path connected space X, we can denote $\pi_1(X, x)$ simply by $\pi_1(X)$ ignoring the mention of the base point x, and call it the fundamental group of the space X.

Since the isomorphism P_{ω} depends on the path ω joining x_0 with x_1, we can examine the question as to how much P_{ω} depends on the path ω itself. We have.

Proposition 3.4.17. If ω, ω' are two paths joining $x_0\, to\, x_1$ which are path homotopic, then the induced isomorphisms P_{ω} and $P_{\omega'}$ are identical.

Proof. If ω, ω' are path homotopic, then it is easily seen that ω^{-1} and $(\omega')^{-1}$ are also path homotopic. It follows that for any loop a based at $x_0, \omega^{-1} * \alpha$ is path homotopic to $(\omega')^{-1} * \alpha$ and therefore $\omega^{-1} * \alpha * \omega$ is path homotopic to $(\omega')^{-1} * \alpha * \omega'$. this means $P_{\omega}[\alpha] = P_{\omega'}[\alpha]$.

Proposition 3.4.18. Let X be a path connected space and $x_0, x_1 \in X$. Then $\pi_1(X, x_0)$ is abelion if and only it for each pair of paths ω, ω' from $P_\omega = P_{\omega'}$.

Proof. Assume that $\pi_1(X, x_0)$ is abelian. Since $\omega * (\omega')^{-1}$ is a loop based at x_0 we observe that for each $[\alpha] \in \pi_1(X, x_0)$,

$$[\omega * (\omega')^{-1}] \circ [\alpha] \circ [\omega * (\omega')^{-1}],$$

Which means

$$[(\omega')^{-1}] * \alpha * \omega'] = [\omega * (\omega')^{-1}],$$

Conversely, suppose $[\alpha], [\beta]$ are two elements of $\pi_1(X, x_0)$, Let ω be a path in X joining x_0 with x_1. Then $\beta * \omega$ is also a path joining x_0 with x_1 .Hence by the given condition. $P_{\beta*\omega}[\alpha] = P_\omega[\alpha]$. This means.

$$[(\beta * \omega)^{-1} * \alpha * (\beta * \omega)] = [\omega^{-1} * \alpha * \omega].$$

Since $(\beta * \omega)^{-1} = \omega^{-1} * \beta^{-1}$,therefore, $(\beta^{-1} * \alpha * \beta] = [\alpha]$, i.e., $[\alpha][\beta] = [\beta][\alpha]$.

Note that to each pointed topological space (X, x), we have associated its fundamental group $\pi_1(X, x)$. Next, we show that for every continuous map $f : (X, x) \to (Y, y)$ of pointed spaces, there is an induced group homomorphism f between their fundamental groups.

Theorem3.4.19.Every continuous map $f : (X, x) \to (Y, y)$ of pointed spaces induces a group homomorphism $f_\# : \pi_1(X, x) \to \pi_1(Y, y)$

Proof. Let α be a loop in X based at x . Then $f \circ \alpha$ is a loop based at y. More over if $\alpha R_{xo} \alpha'$, say, by a homotopy H, then one can easily see that $f \circ \alpha R_y f \circ \alpha'$ *if* $\alpha R_{xo} \alpha'$, say, by the homotopy H. Hence, we define a map $f_\# : \pi_1(X, x) \to \pi_1(Y, y)$ by

$$f_\# : ([\alpha]) = [f \circ \alpha].$$

To we that $f_{\#}$ is a homomorphism, observe that for any two loops α, β based at x we have

$$(f \circ (\alpha * \beta))(t) = f((\alpha * \beta)(t))$$

$$= \begin{cases} f(\alpha(2t)) & 0 \le t \le 1/2 \\ f(\beta(2t-1)) & 1/2 \le t \le 1 \end{cases}$$

$$= \begin{cases} (f \circ \alpha)(2t) & 0 \le t \le 1/2 \\ (f \circ \beta)(2t-1) & 1/2 \le t \le 1 \end{cases}$$

$$= ((f \circ \alpha) * (f \circ \beta))(t)$$

For each $t \in I$, which means $f \circ (\alpha * \beta) = (f \circ \alpha) * (f \circ \beta)$. Hence

$$f_{\#}([\alpha] \circ [\beta]) = f_{\#}[\alpha * \beta]$$

$$= f \circ (\alpha * \beta)]$$

$$= [(f \circ \alpha) * (f \circ \beta)]$$

$$= f_{\#}[\alpha] \circ f_{\#}[\beta].$$

Check Your Progress:

1. Prove: Theorem :Let Then the relation of “being equivalent” in the set of all paths starting from and terminating at is an equivalence relation.

---

2. Prove : Theorem: The set of all path homotopy classes of loops based at is a group with respect to the binary operation define above.

Notes

3. Prove: Theorem: Let be a path connected space and be any two points of . Then and are isomorphic. In fact, each path joining defines on isomorphism from

3.4 LET US SUM UP

Let $x_0, x_1 \in X$. Then the relation of "being equivalent" in the set of all paths starting from x_0 and terminating at x_1 is an equivalence relation.

The set $\pi_1(X, x_0)$ of all path homotopy classes of loops based at x_0 is a group with respect to the binary operation "$\circ$" define above.

Let X be a path connected space and x_0, x_1 be any two points of X. Then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic. In fact, each path joining $x_0 to x_1$ defines on isomorphism from $\pi_1(X, x_0) to \pi_1(X, x_1)$.

If ω, ω' are two paths joining x_0 *to* x_1 which are path homotopic, then the induced isomorphisms P_ω and $P_{\omega'}$ are identical.

Every continuous map $f : (X, x) \to (Y, y)$ of pointed spaces induces a group homomorphism $f_\# : \pi_1(X, x) \to \pi_1(Y, y)$.

3.5 KEY WORDS

Path in a topological space

Homotopy classes of loops

Fundamental group

A group homomorphism

Isomorphism's

3.6 QUESTIONS FOR REVIEW

1. Every continuous map $f:(X,x)\to(Y,y)$ of pointed spaces induces a group homomorphism $f_{\#}:\pi_1(X,x)\to\pi_1(Y,y)$
2. Explain about product of two paths in topological spaces
3. Explain about properties of fundamental group

3.7 SUGGESTIVE READINGS AND REFERENCES

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3.8 ANSWERS TO CHECK YOUR PROGRESS

1. See sub section 3.4.2
2. See sub section 3.4.7
3. See sub section 3.4.15

UNIT-4 CW-COMPLEXES

STRUCTURE

4.0 OBJECTIVE

In this unit we will learn and understand about definitions of CW-complex, operations on CW- complexes, examples of CW-complexes, CW-structure and related theorems.

4.1 INTRODUCTION

Algebraic topologists rarely study arbitrary topological spaces: there is not much one can prove about an abstract topological space. However, there is very well-developed area known as general topology which studies simple properties. (such as connectivity, the Hausdorff property, compactness and so on) of complicated spaces. There is gaint Zoo out there very complicated spaces endowed with all possible degrees of pathology i.e., when one or another simple property fails or holds.

Some of these spaces are extremely useful, such as the Cantor set or fractals, they help us to understand very delicate phenomenas observed in mathematics and physics. In algebraic topology we mostly study complicated properties of simple spaces.

In turns out that the most important spaces which are important for mathematics have some additional structures. The first algebraic topologist, Pincare, studied mostly the spaces endowed with "analytic" structures, i.e., when a space X has natural differential structure of Riemannian metric and so on.
The major advantage of these structures is that they all are natural, so we should not really care about their existence: they are given! There is the other type of natural, so we should not really care about their existence: they are given! There is the other type of natural structures on topological spaces: so called combinatorial structures, i.e, when a space X comes equipped with a decomposition into more or less 'standard pieces', so that one could study the whole space X by examination the mutual geometric and algebraic relations between those "standard pieces".
Below we formalize this concept: these spaces are known as CW-complexes.

4.2 BASIC DEFINITIONS

We will call an open disk D^n (as well as any space homeomorphic to D^n) by $n-cell$. Notation: e^n. We will use the notation e^{-n} for a "closed cell" which is homeomorphic to D^n. For $n=0$ we let $e^0 = D^0$ (point). Let δe^n be a "boundary" of the cell

$e^n : \delta e^n$ is homeomorphic to the sphere S^{n-1}. Recall that if we have a map S^{n-1} then we can construct the space S^{n-1} such that the diagram.

$$\begin{array}{ccc} \bar{e}^n & \xrightarrow{\Phi} & K \cup_\varphi e^n \\ \uparrow i & & \uparrow i \\ \partial e^n & \xrightarrow{\varphi} & K \end{array}$$

Commutes. We will call this procedure on attaching of the cell e^n to the space K. The map $\varphi : \delta e^n \to K$ is the attaching map, and the map $\phi : e^n \to K\, u_\varphi e^n$ the characteristic map of the cell e^n. Notice that ϕ is a homeomorphism of the open cell e^n on its image.

An example of this construction is the diagram (10), where the maps $e : S^n \to RP^n$ and $h : S^{2k+1} \to CP^n$ are the attaching maps of the corresponding cells $e^{n+1} and\ e^{2n+2}$.As we shall see below,

$$RP^n \cup_c e^{n+1} \cong RP^{n+1}\ and\ CP^n \cup_k e^{n+2} \cong CP^{n+1}$$

We return to this particular construction a bit later.

Definition 4.1. A Hausdroff topological space X is a CW computer (or cell-complex)if it is decomposed as a union of cells:

$$X = \bigcup_{q=0}^{\infty} \left(\bigcup_{x \in I_g} e_i^q \right),$$

Where the cells $e_i^q \cap e_j^p = \theta\, unless\, q = p, i = j, and\ for\, each\, e_i^q$ there exists a characteristic map $\phi : D^q \to X$ such that its restriction $\phi_{\overset{\circ}{D}{}^q}$ gives a homeomorphism $\phi\Big|_{\overset{\circ}{D}{}^q} : \overset{\circ q}{D} \to e_i^q$ .It is required that the following axioms are satisfied.

(C) (Close finite): The boundary $\delta e_i^q = e_i^{-q} \setminus e_i^q\ of\ the\, cell\, e_i^q$ is a subset of the union of finite number of cells $e_j^r, where\, r < q.$

(W) (Weak topology): A set $F \subset X$ is closed if and only if the intersection $F \cap e^{-n}$ is closed for every cell e_i^q.

Notes

Example 1. The sphere S^n. There are two standard cell decompositions of S^n

(a) Let e^0 be a point (say, the north pole (0,0,...,0,1) and $e^n = S^n \setminus e^0, so\, S^n e^0 \bigcup e^n$. A characteristic map $D^n \to S^n$ which corresponds to the cell e^n may be defined by $(x_1, x_2, ..., x_n) \to (x_1 \sin \pi p, ..., x_n \sin \pi p, \cos \pi p)$, where $p = \sqrt{x_1^2 + ... + x_n^2}$

(b) We define $S^n = \bigcup_{q=0}^n e_\pm^q$, where $e_\pm^q = \left\{ (x_1, ... x_{n+1}) \in S^n \middle| x_{q+2} = ... = x_{n+1} = 0, and \pm x_{q+1} > 0 \right\}$, see Fig. 11

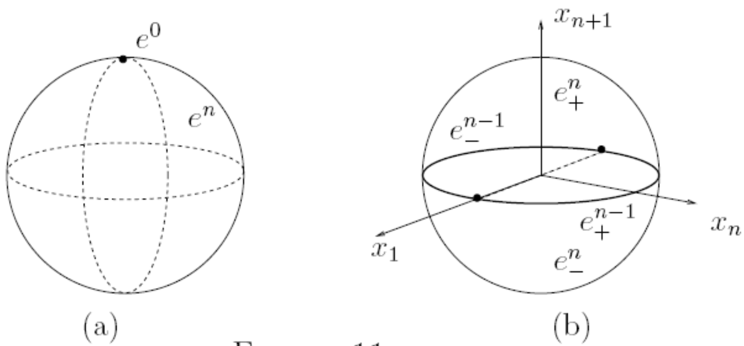


FIGURE 11

There exist a lot more cell decompositions of the sphere S^n: one can decompose S^n on $(3^{n+1} - 1)$ cells as a boundary of $(n+1)$ dimension simplex[4] Δ^{n+1} or on $(2^{n+2} - 2)$ cells as a boundary of the cube I^n.

Exercise 4.1. Describe these cell decompositions of S^n.

Example 2. Any of the above cell decompositions of the sphere S^{n-1} may be used to construct a cell decomposition of the disk D^n by adding one more cell $Id : D^n \to D^n$. The most simple one gives us three cells.

[4] A simplex Δ^k determined as follows:

$$\Delta^k = \left\{ (x_1, ..., x_{k+1}) \in R^{k+1} \middle| x_1 \geq 0, ..., x_{k+1} \geq 0, \sum_{i=1}^{k+1} x_i = 1 \right\}.$$

4.3. COMMENTS ON THE DEFINITION OF A CW-COMPLEX

1^0 Let X be a CW-complex. We donate $X^{(n)}$ the union of all cells in X of dimension $\leq n$. This is the $n-th$ skeleton of X. The $n-th$

skeleton $X^{(n)}$ is an example (very important one) of a subcomplex of a CW-complex. A subcomplex $A \subset X$ is a closed subset of A which is a union of some cells of X .In particular, the $n-th$ skeleton $A^{(n)}$ is a subcomplex of $X^{(n)}$ for each $n \geq 0$ .A map $f : X \to Y$ of CW-complex is a cellular map if $f \left| X^{(n^i)} \right.$ maps the $n-th$ skeleton to the n-skeleton $Y^{(n)}$ for each $n \geq 0$ .In particular, the inclusion $A \subset X$ of a subcomplex is a cellular map. ACW –complex is called finite if it has a finite number of cells. ACW -complex is called locally finite if X has a finite number of cells in each dimension. Finally (X, x_0) is a pointed CW-complex, x_0 is a 0 cell.

Exercise 4.2: Prove that a CW-complex compact if and only if it is finite.

2^0 It turns out that a closure of a cell within a CW-complex may be not a CW-complex.

Exercise 4.3: Construct a cellular decomposion of the wedge $X = S^1 \vee S^2$ (with a single 2-cell e^2) is not a CW subcomplex of X .

The axiom (W) does not imply the axiom (C). Indeed, consider a decomposition of the disk D^2 into 2-cell e^2 which is the interior of the disk D^2 and each point the circle S^1 is considered as a zero cell.

Exercise 4.4. Prove that the disk D^2 with the cellular decomposition described above satisfies (W), and does not satisfy (C).

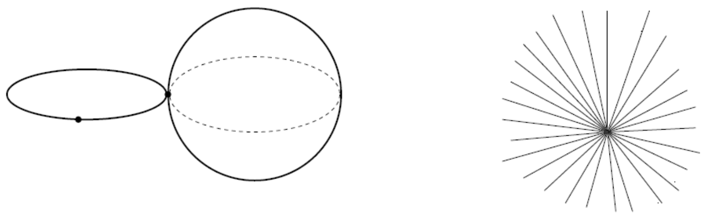


FIGURE 12

The axiom (C). does not imply the axiom (W). Indeed, consider the following space X. We start with an infinite (even countable) family I_α of unit intervals. Let $X = \vee_\alpha I_\alpha$, and $t'' \in I_{\alpha''}$. Then a distance is defined by

$$p(t', t'') = \begin{cases} \left| \ell' - t'' \right| & if\ \alpha' = \alpha'' \\ t' + t'' \quad if & \alpha' \neq \alpha'' \end{cases}$$

Notes

Exercise 4.5. Check that a natural cellular decomposition of X into the interior of I_α and remaining points (zero cells) does not satisfy the axiom (W).

4.4 OPERATIONS ON CW-COMPLEXES

All operations we considered are well-defined on the category of $CW-$ complexes, however we have to be a bit careful. If one of the $CW-$ complexes $X \text{ and } Y$ is locally finite, then the product $X \wedge Y$ of pointed $CW-$complexes. The cone $C(X)$, cylinder $X \times I$, and suspension $\sum(X)$ has canonical $CW-$structure determined by X .We can glue $CW-$complexes $X \bigcup_f Y \, if \, f : A \to Y$ a cellular map, and $A \subset X$ is a subcomplex. Also the quotient space X / A is a $CW-$ complex if (A, X) is a $CW-$pair. The functional spaces $C(X, Y)$ are two bit to have natural $CW-$structure, however, a space $C(X, Y)$ is homotopy equivalent to a $CW-$complex if $X \text{ and } Y$ are $CW-$complexes. The last statement is a nontrivial result due to J. Milnor (1958)

4.5 MORE EXAMPLES OF $CW-$ COMPLEXES

Real projective space RP^n. Here we choose in RP^n. a sequence of projective subspaces.

$$* = RP^0 \subset RP^1 \subset ... \subset RP^{n-1} \subset RP^n.$$

And set $c^0 = RP^0, c^1 = RP^1 \setminus RP^0, ... c^n = RP^n \setminus RP^{n-1}$. The diagram (10) shows that the map $k^{-1} \to RP^k$ is an attaching map, and its extension to the cone over S^{k-1} (the disk D^k) is a characteristic map of the cell e^k. Alternatively this decomposition may be described in the homogeneous coordinates as follows. Let

$$e^q = \left\{(x_0 : x_1 : ... : x_n) \middle| x_q \neq 0, x_{q+1} = 0, ... x_n = 0\right\}.$$

Exercise 4.6. Prove that e^q is homeomorphic to $RP^q \setminus RP^{q-1}$.

Exercise4.7. Construct cell decompositions of $CP^n and\ HP^n$.

Exercise 4.8. Represent as $CW-$complex every 2-dimensional manifold. Try to find a $CW-$into Euclidean space of finite dimension.

4.6 CW-STRUCTURE OF THE GRASSMANIAN MANIFOLDS

We describe here the Schubert decomposition, and the cells of this decomposition are known as the Schubert cells. We consider the space $G(n,k)$. We choose the standard basis $e_1,...,e_n \; of \; R^n .Let\, R^q = \langle e_1,...,e_q \rangle$.

it is convenient to denote $R^0 = \{0\}$. We have the omcisopms"

$$R^0 \subset R^1 \subset R^2 \subset ... \subset R^n.$$

Let $\pi \in G(n,k)$. Clearly π determines a collection of nonnegative numbers

$0 \le \dim(R^1 \bigcap \pi) \le \dim(R^2 \bigcap \pi) + 1.$ Indeed, we have linear maps

$$(11) \qquad 0 \to R^{j-1} \bigcap \pi \xrightarrow{\; i \;} R^j \bigcap \pi \xrightarrow{j-th\,coordinate} R$$

Where the first one, $i : R^{j-1} \bigcap \pi \to R^j \bigcap \pi,$ is an embedding, and the map

$$j-th\,coordinate : R^j \bigcap \pi \to R$$

Is either onto or zero. In the first case dim

$(R^j \bigcap \pi) = \dim(R^{j-1} \bigcap \pi) + 1,$ and in the second case

$(R^j \bigcap \pi) = \dim(R^{j-1} \bigcap \pi).$ Thus there are exactly k "jumps" in the

sequence $(0, \dim(R^1 \bigcap \pi), ..., \dim(R^n \bigcap \pi)).$

A Schubert symbol $\sigma = (\sigma_1, ..., \sigma_n)$ is a collection of integers, such that

$$1 \le \sigma_1 < \sigma_2 < ... < \sigma_k \le n.$$

Let $e(\sigma) \subset G(n,k)$ be the following se of the following

$$k - planes\, in\, R^n$$

$$e(\sigma) = \left\{ \pi \in G(n,k) \middle| \dim(R^\sigma \bigcap \pi) = j \,\&\, \dim(R^{\sigma_{j-1}} \bigcap \pi) = j = 1, j = 1, ..., k \right\}.$$

Notice that every $\pi \in G(n,k)$ belongs to exactly one subset $e(\sigma)$. Indeed, in the sequence of subspaces.

$$R^1 \bigcap \pi \subset R^2 \bigcap \pi \subset ... \subset R^n \bigcap \pi = \pi$$

Their dimensions "jump" by one exactly k times. Clearly $\pi \in e(\sigma), where\, \sigma = (\sigma_1, ..., \sigma_n)$ and

$$\sigma_t = \min\left\{ j \middle| \dim(R^j \cap \pi) = t \right\}.$$

Our goal is to prove that the set $e(\sigma)$ is homeomorphic to an open cell of dimension $d(\sigma) = (\sigma_1 - 1) + (\sigma_2 + -2) + ... + (\sigma_k - k).$ Let $H^j \subset R^n$ denote an open "half $j - plane\, of\ R^j$:

$$H^j = \{(x_1, ..., x_j, 0..., 0) \mid x_j > 0\}.$$

IT will be convenient to denote $\overline{H}^j = \{(x_1, ..., x_j, 0..., 0) \mid x_j > 0\}.$

Claim 4.1. A $k - plane\, \pi$ belongs to $e(\sigma)$ if and only if there exists its basis $\upsilon_1, ..., \upsilon_{k,}$ such that $\upsilon_1 \in H^{\sigma k}.$

Proof. Indeed, if thee is such a basis $\upsilon_1, ..., \upsilon_k$ *then*

$$\dim(R^{\sigma_j} \cap \pi) > \dim(R^{\sigma_{j-1}} \cap \pi)$$

For $j = 1, ..., k.$ Thus $\pi \in e(\sigma).$ The following lemma proves Claim 4.1 in the other direction.

Lemma4.2 Let $\pi \in e(\sigma),$ where $\sigma = (\sigma_1, ..., \sigma_n).$ Then there exists a unique orthonormal basis $\upsilon_1, ..., \upsilon_k of\ \pi, so\, that\, \upsilon_1 \in H^{0\perp}, ..., \upsilon_k \in H^{\sigma k}.$

Proof. We choose υ_1 to be a unit vector which generates the line $R^{\sigma i} \cap \pi.$ There are only two choices here, and the condition that the $\sigma_1 - th$ coordinate it positive determines υ_1 uniquely. Then the unit vector $\upsilon_2 \in R^{\sigma 2} \cap \pi$ should be chosen so that $\upsilon_2 \perp \upsilon_1.$ There are two choices like that, and again the positivity of the $\sigma_2 - th$ coordinate determines υ_2 uniquely. By induction one obtains the required basis. This completes proof of Lemma 4.2 and Claim 4.1.

We define the following subset of the Stiefel manifold $V(n,k)$:

$$E(\sigma) = \left\{(\upsilon_1, ..., \upsilon_k) \in V(n,k) \middle| \upsilon_1 \in H^{\sigma_1}, ..., \upsilon_k \in H^{\sigma_k}\right\}.$$

Lemma 4.2 gives a well-defined map $q : e(\sigma) \to E(\sigma).$ It is convenient to denote $\overline{E}(\sigma)\left\{(\upsilon_1, ..., \upsilon_k) \in V(n,k) \mid \upsilon_1 \in \overline{H}^{\sigma_1}, ..., \upsilon_k \in \overline{H}^{\sigma k}\right\}.$

Claim4.2. The set $\overline{E}(\sigma) \subset V(n,k)$ is homeomorphic to the closed cell of dimension $d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 1) + ... + (\sigma_k - k).$ Furthermore the map $q : e(\sigma) \to E(\sigma)$ is a homeomorphism.

Proof. Induction on k. If $k=1$ the set $\overline{E}(\sigma_1)$ consists of the vectors

$\upsilon_1=(x_{11},...,x_{1\sigma 1},0,...,0),$ such that $\sum x_{1j}^2=1, and\ x_{1\sigma i}\geq 0.$

Clearly $\overline{E}(\sigma_1)$ is a closed hemisphere of dimension $(\sigma_1-1), i.e.\overline{E}(\sigma_1)$ is homeomorphic to the disk $D^{\sigma i-1}$.

To make an induction step, consider the following construction. Let $u,\upsilon\in R^n$ be two unit vectors such that $u\neq -\upsilon. Let\, T_{u,\upsilon}$ an orthogonal transformation $R^n\to R^n$ such that

(1) $T_{\upsilon,v}(u)=\upsilon;\quad T_{\upsilon,v}(\omega)=\omega if\ \omega\langle u,\upsilon\rangle^1.$

In other words, $T_{\upsilon,v}$ is a rotation in the plane $\langle u,\upsilon\rangle$ taking the vector $u\, to\, \upsilon,$ and is identity on the orthogonal complement to the plane $\langle u,\upsilon\rangle$ generated by $u\, to\, \upsilon.$

Claim 4.3. The transformation $T_{\upsilon,v}$ $(where\, u,\upsilon\in R^n, u\neq -\upsilon)$ has the following properties:

$(a)T_{u,u}=Id;$

$(b)T_{u,u}=T_{u,\upsilon}^{-1};$

$(c)T_{u,u}:R^n\to R^n\ is\, be\, given\, by$

$$T_{u,u}(x)=x-\frac{\langle u+\upsilon,x\rangle}{1+\langle u,\upsilon\rangle}(u+\upsilon)+2\langle u,x\rangle\upsilon;$$

$(d)a\, vector\, T_{u,u}(x)$ Depends continuously on $u,\upsilon,x;$

$(e)T_{u,\upsilon}(x)=x(\mathrm{mod}\, R^j)\, if\ u,\upsilon\in R^j.$

The properties (a),(b),(e) follow from the definition.

Exercise 4.10. Prove (c),(d) form Claim 4.3.

Let $Let\ _{\in i}\in H^{\sigma_i}$ be a vector which has σ_i -coordinate equal to 1, and all others are zeros. The $(\in_1,...,\in_k)\in E(\sigma).$For each k-frame $(\upsilon_1,...,\upsilon_k)\overline{E}(\sigma)$ consider the transformation:

$$(12)\quad T=T_{\in k,\upsilon k}\circ T_{\in k-1,\upsilon k-1}\circ T_{\in 1,\upsilon 1}:R^n\to R^n$$

First we notice that $\upsilon_i\neq -\in_i\ \sin ce\, \upsilon_i\in \overline{H}^{\sigma k+1}$:

$$D=\left\{u\in \overline{H}^{\sigma k+1}\ \|u|=1,\ \langle\in_j,u\rangle=0,\ j=1,...,k\right\}.$$

Notes

Exercise 4.11. Prove that the transformation T takes the k-frame $(\in_1,...,\in_k)$ to the frame $(\upsilon_1,...,\upsilon_k)$.

Consider the following subspace $D\subset \overline{H}^{\sigma k+1}$:

$$D=\left\{u\in \overline{H}^{\sigma k+1} \mid |u|=1, \left\langle \in_j ,u\right\rangle =0, j=1,...,k\right\}.$$

Exercise 4.12. Prove that D is homeomorphic to the hemisphere of the dimension $\sigma_{k+1}-k-1.$

This D is a closed cell of dimension $\sigma_{k+1}-k-1.$ Now we make an induction step to comple a proof of Claim 4.2 We define the map

$f:\overline{E}(\sigma_1,...,\sigma_k)\times D\to \overline{E}(\sigma_1,...,\sigma_k,\sigma_{k+1})$

By the formula $f((\upsilon_1,...,\upsilon_k),u)=(\upsilon_1,...,\upsilon_k,T_u)$ Where T is given by (12). We notice that

$\left\langle \upsilon_i,T_u\right\rangle =\left\langle T\in_i,T_u\right\rangle =\left\langle \in_i,u\right\rangle =0, i=1,...,k,$

And $\left\langle T_u,T_u\right\rangle =\left\langle u,u\right\rangle =1$ by definition of $T\, and \sin ce T\in O(n).$

Exercise 4.13. Recall that $\sigma_k<\sigma_{k+1}.$ Prove that $Tu\in \overline{H}^{\sigma k+1}\, if\, u\in D.$

The inverse map $f^{-1}:\overline{E}(\sigma_1,...,\sigma_k,\sigma_{k+1})\to \overline{E}(\sigma_1,...,\sigma_k)\times D$ is defined by

$\upsilon_j=f^{-1}\upsilon_j, j=1,...,k,$

$u=f^{-1}\upsilon_{k+1}=(T^{-1}\upsilon_{k+1})=T_{\upsilon 1,e1}\circ T_{\upsilon 2,e2}\circ......\circ T_{\upsilon k,ek}(\upsilon_{k+1})\in D.$

Both maps $f\, and\, f^{-1}$ are continuous, thus f is a homeomorphism. This concludes induction step in the proof of Claim 4.2. Lemma 4.2 implies that $e(\sigma_1,...,\sigma_k)$ is homeomorphic to an open cell of dimension $d(\sigma)=(\sigma_1-1)+(\sigma_1-2)+...+(\sigma_k-k).$

Remark: Let $(\upsilon_1,...,\upsilon_k)\in \overline{E}(\sigma)\setminus E(\sigma),$ then the k-plane $\pi=\left\langle \upsilon_1,...,\upsilon_k\right\rangle$ does not belong to $e(\sigma).$ Indeed, it means that at least one vector $\upsilon_j\in R^{\sigma_j-1}=\delta\left(\overline{H}^{\sigma j}\right).$ Thus dim $(R^{\sigma_j-1}\bigcap \pi\geq j, here\, \pi\notin e(\sigma).$

Theorem 4.3. A collection of $\binom{k}{n}$ cells $e(\sigma)$ gives $G(n,k)$ a cell-decomposition.

Proof. We should show that any point x of the boundary of the cell $e(\sigma)$ to see that $q(\overline{e(\sigma)}) = \overline{E}(\sigma)$. Thus we can describe $\pi \in \bar{e}(\sigma) \setminus e(\sigma)$ as a k-plane $\langle \upsilon_1, ..., \upsilon_k \rangle$, where $\upsilon_j \in \overline{H}^{\sigma j}$ .Clearly $\upsilon_j \in R^{\sigma j}$,thus dim $(R^{\sigma j} \bigcap \pi) \geq j$ for each $j = 1, ..., k$. Hence $\tau_1 \leq \sigma_1, ..., \tau_k \leq \sigma_k$. However, at least one vector υ_j belongs to the subspace $R^{\sigma j - 1} = \delta\left(\overline{H}^{\sigma j}\right)$, and corresponding $\tau_j < \sigma_j$. Thus $d(\tau) < d(\sigma)$. The number of all cells is equal to $\begin{pmatrix} k \\ n \end{pmatrix}$ by counting. □

Now we count a number of cells of dimension r in the cell decomposition of $G(n,k)$. Recall that a partition of an integer r is an unordered collection $(i_1, ..., i_s)$ such that $i_1 + ... + i_s = r$. Let $p(r)$ be a number of partitions of r .This are values of $p(r)$ for $r \leq 10$.

r	0	1	2	3	4	5	6	7	8	9	10
$p(r)$	1	1	2	3	5	7	11	15	22	30	42

Each Schubert symbol $\sigma = (\sigma_1, ..., \sigma_k)$ of dimension $d = (\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + ... + (\sigma_k - k) = r$ which is given by deleting zeros from the sequence $(\sigma_1 - 1), (\sigma_1 - 2), ..., (\sigma_k - k)$.

Exercise 4.14. Show that

$$1 \leq i_1 \leq i_2 \leq ... \leq i_s \leq k, \quad and\, s \leq n - k.$$

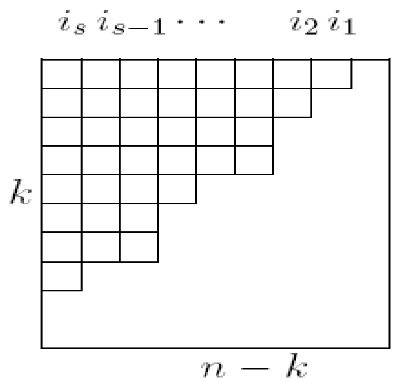
Prove that a number of r-dimensional cells of $G(n,k)$ is equal to a number of partitions $(i_1, ..., i_s)$ of r such that $s \leq n - k\, and\, i_t \leq k$.

Remark. There is a natural chain of embeddings $G(n,k) \to G(n+1,k) \to ... \to G(n+l,k)$. It is easy to notice that these embeddings preserve the Schubert cell decomposition, and if $l\, and\, k$ are large enough, the number of cells of dimension $r\, is\, equal\, to\, p(r)$. In particular, the Schubert, cells give a cell decomposition of $G(\infty,k)\, and\, G(\infty,\infty)$.

Remark. $Let\, \iota = (i_1, ..., i_s)$ be a partition of r as above $(i.e.\, s \leq n - k\, and\, 1 \leq i_1 \leq ... \leq i_s \leq k)$. The partition ι may be represented as a Young tableau.

Notes

This Young tableau gives a parametrization of the corresponding cell $e(\sigma)$. Clearly the Schubert symbols σ are in one-to-one correspondence with the Young tableaux corresponding to the partitions $(i_1,...,i_s)$ as above. The Young tableaus were invented in the representation theory of the symmetric group S_n. This is not an accident, it turns out that there is a deep relationship between the Grasmannian manifolds and the representation theory of the symmetric groups.



Check your progress :

1. Prove: A $k-plane\,\pi$ belongs to $e(\sigma)$ if and only if there exists its basis $\upsilon_1,...,\upsilon_{k,}$ such that $\upsilon_1 \in H^{\sigma k}$.

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2. Prove: The set $\overline{E}(\sigma) \subset V(n,k)$ is homeomorphic to the closed cell of dimension $d(\sigma)=(\sigma_1-1)+(\sigma_2-1)+...+(\sigma_k-k)$. Furthermore the map $q:e(\sigma)\to E(\sigma)$ is a homeomorphism.

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3. Prove: A collection of $\binom{k}{n}$ cells $e(\sigma)$ gives $G(n,k)$ a cell-decomposition

4.7 LET US SUM UP

A Hausdroff topological space X is a CW computer (or cell-complex)if it is decomposed as a union of cells: $X = \bigcup_{q=0}^{\infty}\left(\bigcup_{x\in I_g} e_i^q\right)$,

A set $F \subset X$ is closed if and only if the intersection $F \cap e^{-n}$ is closed for every cell e_i^q.

ACW -complex is called locally finite if X has a finite number of cells in each dimension. Finally (X, x_0) is a pointed CW -complex, x_0 is a 0 cell.

We describe here the Schubert decomposition, and the cells of this decomposition are known as the Schubert cells.

Let X be a CW -complex and $A \subset X$ be its sub complex. Then X / A is homotopy equivalent to the complex $X \cup C(A), Where\, C(A)$ is a cone over $A.$

Any continuous map $f : X \to Y\ of\ CW$ -complexes is homotopic to a cellular map.

A finite triangulation of a subset $X \subset R^n$ is a finite covering of X by simplices $\left\{\Delta^n(i)\right\}$ such that each intersection $\Delta^n(i) \cap \Delta^n(j)$ either emplty, or $\Delta^n(i) \cap \Delta^n(j) = \Delta^{n-1}(i)_k$

For some $k = 0, ..., n.$

4.8 KEY WORDS

Hausdroff topological space

Notes

Borsuk pair

Simplicial complexes

Definitions of the CW-Complexes

CW-Structure

Homotopy

4.9 QUESTIONS FOR REVIEW

1. Write the definitions of CW- Complexes with examples.
2. Explain about Structure of Grassmanian manifolds.
3. Prove: A collection of $\binom{k}{n}$ cells $e(\sigma)$ gives $G(n,k)$ a cell decomposition.

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4.11 ANSWERS TO CHECK YOUR PROGRESS

1. See Claim 4.1
2. See Claim 4.2
3. See Theorem 4.3

UNIT-5 CW-HOMOTOPY

STRUCTURE

5.0 OBJECTIVE

In this unit we will learn and understand about CW- complexes and homotopy, Borsuk’s Theorem on extension of homotopy, Cellular approximation theorem, definitions and theorems.

5.1 INTRODUCTION

In topology, a CW complex is a type of topological space introduced by J. H. C. Whitehead to meet the needs of homotopy theory. This class of spaces is broader and has some better categorical properties than simplicial complexes, but still retains a combinatorial nature that allows for computation (often with a much smaller complex).

Roughly speaking, a CW complex is made of basic building blocks called cells. The precise definition prescribes how the cells may be topologically glued together. The C stands for "closure-finite", and the W for "weak" topology.

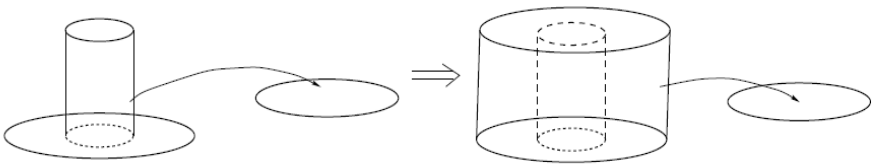
An n-dimensional closed cell is the image of an n-dimensional closed ball under an attaching map. For example, a simplex is a closed cell, and more generally, a convex polytope is a closed cell. An n-dimensional open cell is a topological space that is homeomorphic to the n-dimensional open ball. A 0-dimensional open (and closed) cell is a singleton space. Closure-finite means that each closed cell is covered by a finite union of open cells.

A CW complex is a Hausdorff space X together with a partition of X into open cells (of perhaps varying dimension) that satisfies two additional properties:

- For each n-dimensional open cell C in the partition of X, there exists a continuous map f from the n-dimensional closed ball to X such that
  - the restriction of f to the interior of the closed ball is a homeomorphism onto the cell C, and
  - the image of the boundary of the closed ball is contained in the union of a finite number of elements of the partition, each having cell dimension less than n.
- A subset of X is closed if and only if it meets the closure of each cell in a closed set.

5.2 BORSUK'S THEOREM ON EXTENSION OF HOMOTOPY

We call a pair (of topological spaces) (X, A) a Borsuk pair, if for any map $F : X \to Y$ a homotopy $f_t : A \to Y, 0 \le t \le 1,$ such that $f_0 = F /_A$ may be extended up to homotopy $F_t : X \to Y, 0 \le t \le 1,$ such that $F_t \setminus_A = f_t and\ F_0 = F.$



Figure

A major technical result of this subsection is the following theorem.

Notes

Theorem : (Borsuk) A pair (X, A) *of* CW – complexes is a Borsuk pair.

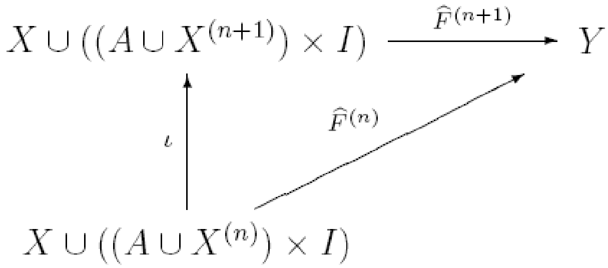
Proof. We are given a map $\phi : A\times 1 \to Y$ (a homotopy f_t) and a map $F : X\times\{0\} \to Y$, such that $F\big|_{A\times\{0\}} = \phi\big|_{A\times\{0\}}$. We combine the maps F *and* ϕ to obtain a map

$F^{'} : X \bigcup (A\times 1) \to Y.$

Where we identify $A \subset X$ *and* $A\times\{0\} \subset A\times I$. To extend a homotopy f_t up to homotopy F_t is the same as to construct a map $F : X\times I \to Y$ such that $F\big|_{X\bigcup(A\times I)} = F$. We construct F by induction on dimension of cells of $X \setminus A$. In more detail, we will construct maps

$F^{(n)} : X \bigcup ((A \bigcup X^{(n)}) \times I) \to Y$

fo each n=0,1,..such that $F^{(n)}\big|_{X\bigcup(A\times I}) = F^{'}$. Furthermore, the following diagram will commute



Where ι is induced by the imbedding $X^{(n)} \bigcup X^{(n+1)}$.

The first step is to extend $F^{'}$ to the space $X \bigcup (A \bigcup X^{(0)}) \times I$ as follows:

$$F^{(o)}(x,t) = \begin{cases} F(x), & \textit{if } x \textit{ is a } 0-\textit{cell from } X \textit{ and if } x \notin A \\ \phi(x,t), & \textit{if } x \in A. \end{cases}$$

Now assume by induction that $F^{(n)}$ is defined on $X \bigcup ((A \bigcup X^{(n)}) \times I)$.

We notice that it is enough to define a map.

$$F^{(n+1)} : X \bigcup ((A \bigcup X^{(n)} \bigcup e^{n+1}) \times I) \to Y$$

Extending $F^{(n)}$ to a single cell

e^{n+1}*, Let* e^{n+1} *be a* $(n+1)-$*cell such that* $e^{n+1} \subset X \setminus A.$

By induction, the map $F^{(n)}$ is already given on the cylinder $(e^{-n+1} \setminus e^{n+1}) \times I$ since the boundary

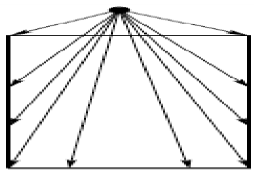
$\delta e^{n+1} = e^{n+1} \subset X^{(n)}$. *Let* $g : D^{n+1} \to X^{(n+1)}$ be a characteristic map corresponding to the cell e^{n+1}. We have to define an extension of $F_1^{(n)}$ from the side $g(S^n)\times I$ and the bottom base $g(D^n)\times\{0\}$ to the cylinder $g(D^{n+1})\times I.$ By definition of $CW-$complex, it is the same as to construct an extension of the map

$$\psi = F^{(n)}\circ g:(D^{n+1}\times\{0\})\bigcup(S^n\times I)\to Y$$

To a map of the cylinder $\psi^{'}:D^{n+1}\times I\to Y.Let$

$$\eta: D^{n+1}\times I\to(D^{n+1}\times\{0\})\bigcup(S^n\times I)$$

Be a projection map of the cylinder $D^{n+1}\times I$ from a point s which is near and a bit above of the top side $D^{n+1}\times\{1\}$ of the cylinder $D^{n+1}\times I$,see the Figure below.



The map η is an identical map on $(D^{n+1}\times\{0\})\bigcup(S^n\times I).$ We define an extension $\psi^{'}$ as follows:

$$\psi^{'}:D^{n+1}\times I\xrightarrow{\eta}(D^{n+1}\times\{0\})\bigcup(S^n\times I)\xrightarrow{\psi}Y.$$

This procedure may be carried out independently for all $(n+1).$ cells of X, so we obtain an extension

$$F^{(n+1)}:X\bigcup((A\bigcup X^{(n+1)})\times I)\to Y.$$

Exercise :Let $D^{n+1}\times I\subset R^{n+1}$ given by:

$$D^{n+1}\times I=\left\{(x_1,...,x_{n+1,}x_{n+2})\middle|x_1^2+...+x_{n+1}^2\le 1,\ x_{n+2}\in[0]\right\}.$$

Give a formula for the above map η.

Thus, going from the skeleton $X^{(n)}$ to the skeleton $X^{(n+1)}$,we construct an extension $F:X\times I\to Y$ of the map $F:X\bigcup(A\times I)\to Y.$

We should emphasize that if X is an infinite-dimensional complex, then our construction consists of infinite number of steps; in that case the axiom (W) implies that F is a continuous map.

Corollary : Let X be a CW-complex and $A\subset X$ be its contractible sub complex. Then X is homotopy equivalent to the complex $X/A.$

Notes

Proof. Let $p: X \to X / A$ be a projection map. Since A is a contractible there exists a homotopy $f_t : A \to A$ such that $f_0 : A \to A$ is an identity map, and $f_1(A) = x_0 \in A.$ By Theorem 5.1 there exists a homotopy $f_1 : A \to A$ such that $f_0 : A \to A$ is an identity map, and $f_1(A) = x_0 \in A.$ By Theorem 4.8 there exists a homotopy $F_t : X \to X, 0 \le t \le 1,$ such that $F_0 = Id_X$ and $F_t |_A = f_t.$ In particular, $F_1(A) = x_0.$ It means that F_1 may be considered as a map given on $X / A,$ (by definition o f the quotient topology),i.e.

$$F_1 = q \circ p : X \xrightarrow{p} X / A \xrightarrow{q} X,$$

Where $q : X / A \to X$ is some continuous map. By construction. $F_1 R F_0, i.e. q \circ p R Id_X$ ..Now, $F_1(A) \subset A$ for any t, i.e. $p \circ F_t(A) = x_0.$ It follows that $p \circ F_t = h_t \circ p,$ where $h_t : X / A \to X / A$ is some homotopy, such that $h_0 = Id_{X/A} \, and \, h_1 = p \circ q;$ it means that $p \circ q R Id_{X/A}.$

Corollary :Let X be a CW-complex and $A \subset X$ be its sub complex. Then X / A is homotopy equivalent to the complex $X \cup C(A), Where \, C(A)$ is a cone over $A.$

5.3 CELLULAR APPROXIMATION THEOREM

Let $X \, and \, Y$ be CW-complexes. Recall that t map $f : X \to Y$ is a cellular map if $f(X^{(n)}) \subset Y^{(n)}$ for every $n = 0,1,....$ We emphasize that it is not required that the image of $n-cell$ belongs to a union of $n-cells$. For example, a constant map $* : X \to x_0 = e^0$ is a cell map. The following theorem provides very important tool in algebraic topology.

Theorem 5.1 Any continuous map $f : X \to Y \, of \, CW$-complexes is homotopic to a cellular map.

We shall prove the following stronger statement:

Theorem 5.2. Let $f : X \to Y$ be a continuous map of CW-complexes, such that a restriction f / A is a cellular map on a CW-sub complex

$A \subset X$. Then there exists a cell map $g: X \to Y$ such that $g/_A = F/_A$ and, moreover, $fg \in g \, rel \, A$.

First of all, we should explain the notation $f \in g \, rel \, A$.

First of all, we should explain the notation $f \in g \, rel \, A$ which we are using. Assume that we are given two maps $f, g: X \to Y$ such that $f|_A = g|_{A.}$ A notation $f \in g \, rel \, A$ means that there exists a homotopy $h_t: X \to Y$ such that $h_t(a)$ does not depend on t for any $a \in A$.

Certainly $f \in g \, rel \, A$ implies $f \in g$,but $f \in g$ does not imply $f \in g \, rel \, A$.

Exercise :Give an example of a map $f:[0,1] \to S^1$ which is homotopic to a constant, map, and, at the same time f is not homotopic to a constant map relatively to $A = \{0\} \cup \{1\} \subset I$.

Proof of Theorem 5.2 We assume that f is already a cellular map not only on A, but also on all cells of X of dimension less or equal to $(p-1)$. consider a cell $e^p \subset X \setminus A$. The image $f(e^p)$ has nonempty intersection only with a finite number of cells of Y: this is because $f(e^{-p})$ is a compact. We choose a cell of maximal dimension $e^q of \, Y$ such that it has nonempty intersection with $f(e^p)$. If $q \le p$, then we are done with the cell e^p and we move to another $p-cell$. Consider the case when $p > p$. Here we need the following lemma.

Lemma :(Free-point-Lemma) Let U be an open subset of R^p, and $\varphi: U \to D^{\circ q}$ be a continuous map such that the set $V = \varphi^{-1}(d^\circ) \subset U$ is compact for some closed disk $d^q \subset D^{0q}$. If $q > p$ there exists a continuous map $\psi: U \to D^{\circ q}$ such that

1. $\psi | U \setminus v = \varphi | u \setminus v$;

2. The image $\psi(V)$ does not cover all disk d'',i.e. there exists a point $yo = d'' \setminus \psi(U)$.

We postpone a proof of this Lemma for a while.

Notes

Remark: The maps $\varphi\, and\, \psi$ from Lemma 5.6 are homotopic relatively to $U\setminus V$: it is enough to make a linear homotopy: $h_1(x)=(1-t)\varphi(x)+t\psi(x)$ since the disk $D^{\circ q}$ is a convex set.

Claim 5.1 Lemma 5.4 implies the following statement: The map

$f\,|_{AUX^{(p-1)}Ue^p}$ is homotopic rel $(A\bigcup X^{(p-1)})\,to\,a\,map \quad f^1 : A\bigcup X^{(p-1)}\bigcup e^p \to Y$, such that the image $f^1(e^p)$ does not cover all cell e^p.

Proof. Indeed, let $h: D^p \to X, k: D^q \to Y$ be the characteristic maps of the cells $e^p\, and\, e^q$ respectively, Let

$$U = h^{-1}(e^p \bigcap f^{-1}(c^q)),$$

And let $\varphi : U \to D^{\circ q}$ be the composition:

$$U \xrightarrow{h} e^p \bigcap f^{-1}e^q \xrightarrow{f} e^q \xrightarrow{k^{-1}} D^{\circ q}.$$

Let d^q be a small disk inside $D^{\circ q}$ (with the same center as $D^{\circ q}$). The set $V=\varphi^{-1}(d^q)$ is compact (as a closed subset of the disk $D^{\circ p}$). Let $\psi : U \to D^{\circ q}$ be a map from Lemma 4.14. We define a map $f^{'}\, on\, h(U)$ as the composition:

$$h(U) \xrightarrow{h^{-1}} U \xrightarrow{\psi} D^{\circ q} \to \in^q \subset Y,$$

And $f^{'}(x) = f(x)\ for\ x \notin h(U)$. Clearly the map

$$f^{'} : A\bigcup X^{(p-1)} \bigcup e^p \to Y$$

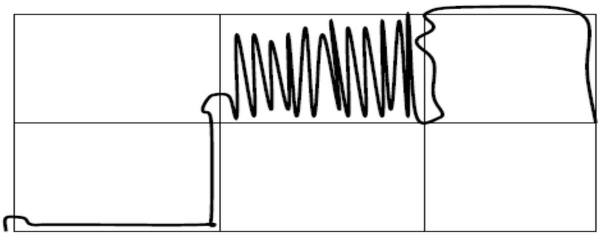
Is continuous (since it coincides with $f\ on\, h(U\setminus V))\, and$

$$f^{'} : A\bigcup X^{(p-1)} \bigcup e^p \to Y \,\square\, f\,|_{A\bigcup X^{(p-1)}\bigcup e^p}\ rel\,(A\bigcup X^{(p-1)}),$$

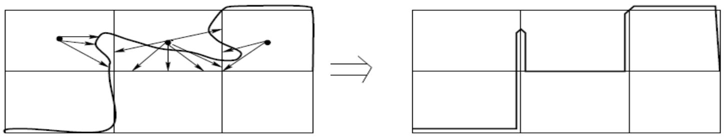
Moreover,

$$f^{'} : A\bigcup X^{(p-1)} \bigcup e^p \to Y \,\square\, f\,|_{A\bigcup X^{(p-1)}\bigcup e^p}\ rel\,(A\bigcup X^{(p-1)} \bigcup (e^p \setminus h(V)))$$

(the latter follows from a homotopy $\varphi \simeq \psi\, rel(U \setminus V)$). Also it is clear that $f^{'}(e^{p})$ does not cover all cell e^{p}.



Figure



Figure

5.1 Completion of the proof of Theorem 5.4 Now the argument is simple . Fistly, ahomotopy between the maps

$$f\,|_{A\bigcup X^{(p-1)}\bigcup e^{p}} \;\; and\; f^{'} \;\; rel\,(A\bigcup X^{(p-1)})$$

Can be extend to all X by Borsuk Theorem. In particular, we can assume that $f^{'}$ with all above properties is defined on all X .

Secondly, we consider a point $y_0 \in \in^{q} \subset Y$ which does not belong to the image $f^{1}(e^{p})$, and "blow away" the map $f^{'}|_{e^{p}}$ from that point as it is shown at Fig. 15. This is a homotopy which may be described as follows.

If $x \in e^{p}, and\; x \notin (f^{'})^{-1}(\in^{q}), then H_t(x) = f^{'}(x)\; for\; all\, t.$

If $x \in e^{p}, and\; x \notin (f^{'})^{-1}(\in^{q}), then\; f^{'}(x)$ moves along the ray connecting y_o and the boundary of $\in^{q}$ to a point on the boundary of $\in^{q}$.

We extend this homotopy to a homotopy of the map $f^{'}|_{A\bigcup X^{(p-1)}\bigcup e^{p}.}$ (relatively to e^{p}), and then up to homotopy the map $f^{'}: X \to Y.$ The resulting map $f^{''}$ is homotopic to $f^{'}(and\; f), and\; f^{''}(e^{p})$ does not touch the cell $\in^q$ and any other cell of dimension $> p$. Now we can apply the procedure just described several times and we obtain a map f_1 homotopic to $f, such that$ f_1 is a cellular map on the subcomplex $A \bigcup X^{(p-1)} \bigcup e^p$. Note that each time we applied homotpoy it was fixed on (relative to) $A \bigcup X^{(p-1)}$. It justifies the induction step, and proves the theorem.

Exercise: Find all pints in the argument from “Completion of the proof of Theorem 5.3” were we have used Borsuk Theorem.

Remark. Again, if the $CW-$complex X is infinite, then the axiom (W) takes care for the resulting cellular map to be continuous.

5.2 fighting a phantom: Proof of Lemma 5.1. There are two well-knwon ways to prove our Lemma. The first one is to approximate our map by a smooth one, and then apply so called Sard Theorem. The second way is to use a simplicial approximation of continuous maps. The first way is more elegant, but the second is elementary, so we prove our Lemma following the second idea. First we need some new “standard spaces” which live happily inside the Euclidian space R^n.

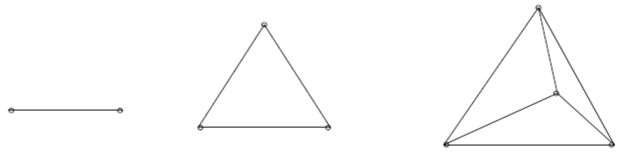
Let $q \leq n+1, and \overrightarrow{\upsilon_1},..,\overrightarrow{\upsilon}_{q+1}$ be vectors those endpoints do not belong to any $(q-1)-$ dimensional subspace. We call the set.

$$\Delta^q(\overline{\upsilon}_1,...,\overline{\upsilon}_{q+1)=}\left\{t_1\overline{\upsilon}_1+...+t_q\overline{\upsilon}_{q+1} \mid t_1+...+t_{q+1}=1, t_1 \geq 0,...,t_{q+1} \geq 0\right\}$$

$a\, q-$ Dimensional simplex.

Exercise : $\Delta^q = \left\{(x_1,...,x_{q+1}) \in R^{q+1} \mid x_1 \geq 0,...,x_{q+1} \geq 0, \sum_{i=1}^{q+1} x_i = 1\right\}$.

Example. A $0-simplex$ is a point; a simplex Δ^1 is the interval connecting two points; a simplex δ^2 is a non degenerated triangle in the space R^2; a simplex Δ^3 is a pyramid in R^n with the vertices $\vec{\upsilon}_0, \vec{\upsilon}_1, \vec{\upsilon}_2, \vec{\upsilon}_3$, see the picture below:



A $j-th$ side of the simplex

$\Delta^q(\vec{\upsilon}_i,...,\vec{\upsilon}_{j-1,}\vec{\upsilon}_{j+1},...,\vec{\upsilon}_q)_j = \left\{t_1\vec{\upsilon}_1 + ... + t_{q+1}\vec{\upsilon}_{q+1} \in \Delta^q(\vec{\upsilon}_1,...,\vec{\upsilon}_{q+1}) \,|\, t_j = 0\right\}.$

We are not going to develop a theory of simplicial complexes this theory is parallel to the theory of $CW-$complexes),however we need the following definition

Definition : A finite triangulation of a subset $X \subset R^n$ is a finite covering of X by simplices $\left\{\Delta^n(i)\right\}$ such that each intersection $\Delta^n(i) \bigcap \Delta^n(j)$ either empty, or

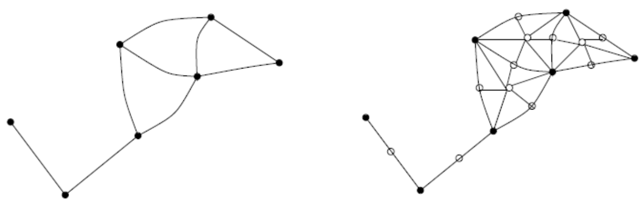
$$\Delta^n(i) \bigcap \Delta^n(j) = \Delta^{n-1}(i)_k$$

For some $k = 0,...,n.$

Exercise: Let $\Delta_1^n,...,\Delta_s^n$ be a finite set of $n-$ dimensional simplexes in R^n. Prove that the union $K = \Delta_1^n \bigcup \Delta_2^n \bigcup ... \bigcup \Delta_s^n$ is a finite simplicial complex.

Exercise: Let $Let \Delta_1^p, \Delta_2^q$ be two simplices. Prove that $K = \Delta_1^p \times \Delta_2^q$ is a finite simplicial complex.

A barycentric subdivision of a q-simplex Δ^q is a subdivision of this simplex on $(g+1)!$ smaller simplices as follows. First let us look at the example:



In general, we can proceed by induction. The picture above shows a barycentric subdivision of the simplices $\Delta^1, and\, \Delta^2$. Assume by

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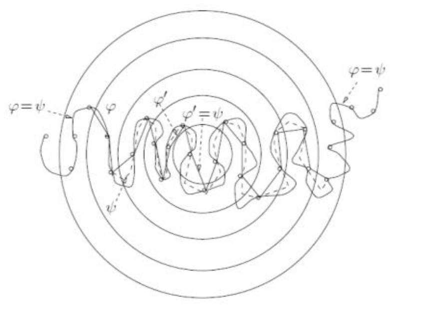
induction that we have defined a barycentric subdivision of the simplices $\Delta^j \, for \; j \le q-1.$ Now let x^* be a weight center of the simplex Δ^q. We already hae a barycentric subdivision of each $j - the \, side \, \Delta_j^q by \, (q-1)$ - simplices $\Delta_j^{(1)}, ..., \Delta_j^{(n)}, n = q!.$ The cones over these simplices, $j = 0, ..., q,$ with a vertex x^* constitute a bary centric subdivision of Δ^q. Now we will prove the following "Approximation that $\Delta^n(i) \subset U$.

Proof. For each point $x \in \overline{V}$ there exists a simplex $\Delta^n(x)$ with a center at $x \, and \, \Delta^n(x) \subset U.$ By compactness of $\overline{V}$ there exist a finite number of simplices $\Delta^n(x_i)$ covering $\overline{V}$ .It remains to use Exercise to conclude that a union of finite number of $\Delta^n(x_i)$ has a finite triangulation .

5.2 Back to the Proof of Lemma 5.4. We consider carefully our map $\varphi : U \to D^{\circ q}$ .First we construct the disks d_1, d_2, d_3, d_4 inside the disk d with the same center and of radii $r/5, 2r/4, 3r/5, 4r/5$ respectively, where r is a radius of d . Then we cover $V = \varphi^{-1}(d)$ by finite number of p-simplexes $\Delta^p(j), such \, that \, \Delta^n(j) \subset U.$ Making, if necessary, a barycentric subdivision (a finite number of times) of these simplices, we can assume that each simplex Δ^p has a diameter $d(\varphi(\Delta^p)) < r/5. Let \, k_1$ be a union o fall simplices Δ^q such that the intersection $\varphi(\Delta) \bigcap d_4$ is not empty. Then

$$d_4 \bigcap \varphi(U) \subset \varphi(K_1) \subset d /$$

Now we consider a map $\varphi^{'} : K_1 \to d_4$ which coincides with φ on all vertices of our triangulation, and is linar on each simplex $\Delta \subset K_1.$ The maps $\varphi|_{K_1} \, and \varphi^{'}$ are homotopic, i.e. there is a homotopy $\varphi_1 : K_1 \to d_4,$, such that $\varphi_0 = \varphi|_{K_1} \; and \; \varphi_1 = \varphi^{'}.$



Exercise. Construct a homotopy φ_t as above.

Now we construct a map $\psi : U \to \overset{\circ q}{D}$ ot of maps $\varphi, \varphi_t and\, \varphi^{'}$ as follows:

$$\psi(u) = \begin{cases} \varphi(u) & if\, \varphi(u) \notin d_3, \\ \varphi^{'}(u) & if\, \varphi(u) \in d_2, \\ \varphi_{3-\frac{5r(u)}{r}(u)} & if\, \varphi(u) \in d_3 \setminus d_2. \end{cases}$$

Here $r(u)$ is a distance from $\varphi(u)$ to a center of the disk $d, see\, Fig.5.7.$

Now we notice that ψ is a continuous map, and it coincides with φ on $U/V.$ Further more,the intersection of its image with $d_{1,} the\, set\, \psi(U) \bigcap d_1,$ is a union of finite number of pieces of p-dimensional planes, i.e. there is a point $y \in d_1 which\, y \notin \psi(U).$

Exercise : Let $\pi_1,...,\pi_s$ be a finite number of p-dimensional planes in $R^q, where\ p < q.$ Prove that the union $\pi_1 \bigcup ... \bigcup \pi_s$ does cover any open subset $\bigcup \subset R^n.$

Thus Cellular Approximation Theorem proved.

First applications of Cellular Approximation Theorem. We start with the following important result.

Theorem : Let $X\, be\, a\, CW$ -complex with only one zero-cell and without q-cells for $0 < q < n, and\, Y$ be a CW -complex of dimension $< n, i.e. Y - Y^{(k)}, where\, k < n.$ Then any map $Y \to X$ is homotopic to a

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constant map. The same statement holds for “pointed”spaces and “pointed” maps.

Remark. For each pointed space (X, x_0) define $\pi_k(X, x_0) = [S^k, X]$ (where we consider homotpy classes of maps $f:(S^k, s_0)) \to (X, x_0))$. Very soon we will learn a lot about $\pi_k(X, x_0)$, in particular, that there is a natural group structure on $\pi_k(X, x_0)$ which are called homotopy group s of X.

The following statement is a particular case of Theorem 5.9:

Corollary: The homotopy groups $\pi_k(S^n)$ are trivial for $1 \le k < n$.

We call a space $X\ n-$connected if it is path-connected and $\pi_k(X) = 0\ for\ k = 1,\ldots,n.$

Exercise : Prove that a space X is 0-connected if and only if it is path-connected.

Check your progress :

1. Prove: A pair $(X, A)\ of\ CW-$ complexes is a Borsuk pair.

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2. Prove: Let X be a CW-complex and $A \subset X$ be its contractible sub complex. Then X is homotopy equivalent to the complex X/A.

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3. Prove: Let $f: X \to Y$ be a continuous map of CW-complexes, such that a restriction f/A is a cellular map on a CW-sub complex $A \subset X$. Then there exists a cell map $g: X \to Y$ such that $g/_A = F/_A$ and, moreover, $fg \in g\ rel\ A$.

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5.4 LET US SUM UP

1. A pair $(X,A)\, of\ CW-$complexes is a Borsuk pair

2. Let X be a CW-complex and $A\subset X$ be its contractible sub complex. Then X is homotopy equivalent to the complex $X/A.$

3. Any continuous map $f: X\rightarrow Y\ of\ CW$-complexes is homotopic to a cellular map.

4. Let $f: X\rightarrow Y$ be a continuous map of CW-complexes, such that a restriction f/A is a cellular map on a CW-sub complex $A\subset X.$ Then there exists a cell map $g: X\rightarrow Y$ such that $g/_A = F/_A$ and, moreover, $fg\in g\ rel\ A.$

5. A finite triangulation of a subset $X\subset R^n$ is a finite covering of X by simplices $\{\Delta^n(i)\}$ such that each intersection $\Delta^n(i)\bigcap\Delta^n(j)$ either empty, or

$\Delta^n(i)\bigcap\Delta^n(j)=\Delta^{n-1}(i)_k$ For some $k=0,\dots,n.$

6. The homotopy groups $\pi_k(S^n)$ are trivial for $1\le k<n.$

We call a space $X\ n-$connected if it is path-connected and $\pi_k(X)=0\ for\ k=1,\dots,n.$

5.5 KEY WORDS

Borsuk pair

Borsuk's theorem

Simplicial complexes

Definitions of the CW-Complexes

CW-Structure

Homotopy

Cellular approximation theorem

5.6 QUESTIONS FOR REVIEW

1. Explain about Borsuk's extention of homotopy
2. Explain about Cellular approximation theorem.
3. Explain about CW-Complex and homotopy

5.7 SUGGESTIVE READINGS AND REFERENCES

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5.8 ANSWERS TO CHECK YOUR PROGRESS

1. See section 5.3
2. See section 5.3
3. See section 5.3
4. See section 5.3

UNIT- 6 INTRODUCTION TO SHEAVES AND THEIR COHOMOLOGY

STRUCTURE

6.0 OBJECTIVE

In this unit we will learn and understand about definitions of Cohomology, Direct and inverse images of pre sheaves and sheaves, Cohomology of sheaves, Comparison with other Cohomologies, Sheaf Cohomology.

6.1 INTRODUCTION

In mathematics, specifically in homology theory and algebraic topology, cohomology is a general term for a sequence of abelian groups associated to a topological space, often defined from a cochain complex. Cohomology can be viewed as a method of assigning richer algebraic invariants to a space than homology. Some versions of cohomology arise by dualizing the construction of homology. In other words, cochains are functions on the group of chains in homology theory.

From its beginning in topology, this idea became a dominant method in the mathematics of the second half of the twentieth century. From the initial idea of homology as a method of constructing algebraic invariants of topological spaces, the range of applications of homology and cohomology theories has spread throughout geometry and algebra. The terminology tends to hide the fact that cohomology, a contravariant theory, is more natural than homology in many applications. At a basic level, this has to do with functions and pullbacks in geometric situations: given spaces X and Y, and some kind of function F on Y, for any mapping $f : X \to Y$, composition with f gives rise to a function $F \circ f$ on X. The most important cohomology theories have a product, the cup product, which gives them a ring structure. Because of this feature, cohomology is usually a stronger invariant than homology.

6.2 DEFINITIONS

Let X be a topological space.

DEFINITION 6.1. A pre sheaf of Abelian groups on X is a rule[1] P which assigns an Abelian group P(U) to each open subset U of X and a morphism (called restriction map) $\varphi_{U,V} : P(U) \to P(V)$ to each pair $V \subset U$ of open subsets, so as to verify the following requirements:

(1) $P(\emptyset) = \{0\}$;

(2) $\varphi_{U,V}$ is the identity map;

(3) if $W \subset V \subset U$ are open sets, then $\varphi_{U,W} = \varphi_{V,W} \circ \varphi_{U,V}$.

The elements $s \in P(U)$ are called sections of the presheaf P on U. If $s \in P(U)$ is a section of P on U and $V \subset U$, we shall write $s_{|V}$ instead of $\varphi_{U,V}(s)$. The restriction $P_{|U}$ of P to an open subset U is defined in the obvious way.

Presheaves of rings are defined in the same way, by requiring that the restriction maps are ring morphisms. If R is a presheaf of rings on X, a presheaf M of Abelian groups on X is called a presheaf of modules over R (or an R-module) if, for each open subset U, M(U) is an R(U)-module and for each pair $V \subset U$, the restriction map $\varphi_{U,V} : M(U) \to M(V)$ is a morphism of R (U)-modules (where M (V) is regarded as an R(U)-module via the restriction morphism $R(U) \to R(V)$). The definitions in this Section are stated for the case of presheaves of Abelian groups, but analogous definitions and properties hold for presheaves of rings and modules.

Definition 6.2. A morphism $f : P \to Q$ of presheaves over X is a family of morphisms of Abelian groups $f_U : P(U) \to Q(U)$ for each open $U \subset X$, commuting with the rather native terminology can be made more precise by saying that a presheaf on X is a contravariant factor from the category O_X of open subsets of X to the category of Abelian groups. O_X is defined as the category whose objects are the open subsets of X while the morphisms are the inclusions of open sets.

restriction morphisms; i.e., the following diagram commutes:

$$\begin{array}{ccc} P(U) & \xrightarrow{fU} & Q(U) \\ {\scriptstyle \varphi U,V}\downarrow & & \downarrow{\scriptstyle \varphi U,V} \\ P(V) & \xrightarrow{fV} & Q(V) \end{array}$$

Definition 6.3. The stalk of a presheaf P at a point $x \in X$ is the Abelian group

$$P_x = \varinjlim_{U} P(U)$$

where U ranges over all open neighbourhoods of x, directed by inclusion.

Remark 6.4. We recall here the notion of direct limit. A directed set I is a partially ordered set such that for each pair of elements $i, j \in I$ there is a third element k such that $i < k$ and $j < k$. If I is a directed set, a directed system of Abelian groups is a family $\{G_i\}_{i\in I}$ of Abelian groups, such that for all $i < j$ there is a group morphism $f_{ij} : G_i \to G_j$, with $f_{ii} = id$ and $f_{ii} \circ f_{jk} = f_{jk} = f_{ik}$. On the set $B = \coprod_{i\in I} G_i$, where $\coprod$ denotes disjoint union, we put the following equivalence relation: $g \sim h$, with $g \in G_i$ and $h \in G_j$, if there exists a $k \in I$ such that $f_{ik}(g) = f_{JK}(h)$. The direct limit l of the system $\{G_i\}_{i\in I}$, denoted $l = \varinjlim_{i\in I} G_i$, is the quotient $B/\sim$. Heuristically, two elements in B represent the same element in the direct limit if they are `eventually equal.' From this definition one naturally obtains the existence of canonical morphisms $G_i \to l$. The following discussion should make this notion clearer; for more detail, the reader may consult [13].

If $x \in U$ and $s \in P(U)$, the image s_x of s in P_x via the canonical projection $P(U) \to P_x$ (see footnote) is called the germ of s at x. From the very definition of direct limit we see that two elements $s \in P(U), s' \in P(V), U, V$ being open neighbourhoods of x, define the same germ at x, i.e. $s_x = s'_x$, if and only if there exists an open neighbourhood $W \subset U \cap V$ of x such that s and s' coincide on $W, s_{|W} = s'_{|W}$.

Definition 6.5. A sheaf on a topological space X is a presheaf F on X which fulfils the following axioms for any open subset U of X and any cover $\{U_i\}$ of U.

S1) If two sections $s \in F(U), \bar{s} \in F(U)$ coincide when restricted to any $U_i, s_{|U_i} = \bar{s}_{|U_i}$, they are equal, $s = \bar{s}$.

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S2) Given sections $s_i \in F(U_i)$ which coincide on the intersections, $s_{i|U_i \cap U_j = s_j|U_i \cap U_j}$ for every i, j, there exists a section $s \in F(u)$ whose restriction to each U_i equals s_i, i.e. $s_{|U_i = s_i}$.

Thus, roughly speaking, sheaves are presheaves defined by local conditions. The stalk of a sheaf is defined as in the case of a presheaf.

Example 6.6. If F is a sheaf, and $F_x = \{0\}$ for all $x \in X$, then F is the zero sheaf, $F(U) = \{0\}$ for all open sets $U \subset X$. Indeed, if $s \in F(U)$, since $s_x = 0$ for all $x \in U$, there is for each $x \in U$ an open neighbourhood U_x such that $s_{|U_x} = 0$. The first sheaf axiom then implies $s = 0$. This is not true for a presheaf, cf. Example 6.15 below.

A morphism of sheaves is just a morphism of presheaves. If $f : F \to G$ is a morphism of sheaves on X, for every $x \in U$ the morphism f induces a morphism between the stalks, $f_x : F_x \to G_x$, in the following way: since the stalk F_x is the direct limit of the groups $F(U)$ over all open U containing x, any $g \in F_x$ is of the form $g = s_x$ for some open $U \ni x$ and some $s \in F(U)$; then set $f_x(g) = (f_U(s))_x$.

A sequence of morphisms of sheaves $0 \to F' \to F \to F'' \to 0$ is exact if for every point $x \in X$, the sequence of morphisms between the stalks $0 \to F_x' \to F_x \to F_x'' \to 0$ is exact. If $0 \to F' \to F \to F'' \to 0$ is an exact sequence of sheaves, for every open subset $U \subset X$ the sequence of groups $0 \to F'(U) \to F(U) \to F''(U)$ is exact, but the last arrow may fail to be surjective. Instances of this situation are shown in Examples 6.11 and 6.12 below.

Exercise 6.7. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves. Show that $0 \to F' \to F \to F''$ is an exact sequence of presheaves.

Example 6.8. Let G be an Abelian group. Defining $P(U) \equiv G$ for every open subset U and taking the identity maps as restriction morphisms, we obtain a presheaf, called the constant presheaf G_X. All stalks $(G_X)_x$ of G_X are isomorphic to the group G. The presheaf G_X is not a sheaf: if

V_1 and V_2 are disjoint open subsets of X, and $U = V_1 \cup V_2$, the sections $g_1 \in G_x(V_1) = G, g_2 \in G_x(V_2) = G,$ with $g_1 \neq g_2$, satisfy the hypothesis of the second sheaf axiom S2) (since $V_1 \cap V_2 = \phi$ there is nothing to satisfy), but there is no section $g \in G_X(U) = G$ which restricts to g_1 on V_1 and to g_2 on V_2.

Example 6.9. Let $C_X(U)$ be the ring of real-valued continuous functions on an open set U of X. Then C_X is a sheaf (with the obvious restriction morphisms), the sheaf of continuous functions on X. The stalk $C_x \equiv (C_X)_x$ at x is the ring of germs of continuous functions at x.

Example 6.10. In the same way one can define the following sheaves:

The sheaf C_X^∞ of differentiable functions on a differentiable manifold X.

The sheaves Ω_X^p of differential p-forms, and all the sheaves of tensor fields on a differentiable manifold X.

The sheaf of holomorphic functions on a complex manifold and the sheaves of holomorphic p-forms on it.

The sheaves of forms of type (p,q) on a complex manifold X.

Example 6.11. Let X be a differentiable manifold, and let $d : \Omega_X^\bullet \to \Omega_X^\bullet$ be the exterior differential. We can define the presheaves Z_X^p of closed differential p-forms, and B_X^p of exact p-differential forms,

$$Z_X^p(U) = \{ w \in \Omega_X^p(U) \mid dw = 0 \},$$

$$B_X^p(U) = \{ w \in \Omega_X^p(U) \mid w = d\tau \quad for \;\; some \;\; \tau \in \Omega_X^p(U) \}.$$

Z_X^p is a sheaf, since the condition of being closed is local: a differential form is closed if and only if it is closed in a neighbourhood of each point of X. On the contrary, B_X^p is not a sheaf. In fact, if $X = R^2$, the presheaf B_X^1 of exact differential 1-forms does not fulfill the second sheaf axiom: consider the form

$$w = \frac{xdy - ydx}{x^2 + y^2}$$

defined on the open subset $U = X - \{(0,0)\}$. Since w is closed on U, there is an open cover $\{U_i\}$ of U by open subsets where w is an exact form, $\omega_{|U_i} \in B_X^1(U_i)$ (this is Poincare's lemma). But ω is not an exact form on U because its integral along the unit circle is different from 0.

This means that, while the sequence of sheaf morphisms $0 \to R \to C_X^\infty \xrightarrow{d} Z_X^1 \to 0$ is exact (Poincare lemma), the morphism $C_X^\infty(U) \xrightarrow{d} Z_X^1(U)$ may fail to be surjective.

Example 6.12. Let X be a complex manifold, R the constant sheaf with stalk the integers, O the sheaf of holomorphic functions on X, and O^* the sheaf of nowhere vanishing holomorphic functions. In analogy with the exact sequence (1.1) we may consider the sequence

(6.1) $$0 \to R \to O \xrightarrow{\exp} O^* \to 1$$

This is an exact sequence of sheaves, in particular exp: $R \to R^*$ is surjective as a map of sheaves, since in a neighbourhood of every point, an inverse my be found by applying the logarithm function. However, since the latter is multi-valued, surjectivity fails on non-simply connected open sets. See Example 6.11.

1.1. Etale space. We wish now to describe how, given a presheaf, one can naturally associate with it a sheaf having the same stalks. As a first step we consider the case of a constant presheaf G_X on a topological space X, where G is an Abelian group. We can define another presheaf G_X on X by putting $G_X(U)$ = {locally constant functions $f : U \to G$ },[2] where $G_X(U) = G$ is included as the constant functions. It is clear that $(G_X)_x = G_x = G$ at each point $x \in X$ and that G_X is a sheaf, called the constant sheaf with stalk G. Notice that the functions $f : U \to G$ are the sections of the projection $\pi : \coprod_{x\in X} G_x \to X$ and the locally constant functions correspond to those sections which locally coincide with the sections produced by the elements of G.

Now, let P be an arbitrary presheaf on X. Consider the disjoint union of the stalks $\underline{P} = \coprod_{x\in X} P_x$ and the natural projection $\pi : \underline{P} \to X$. The sections $s \in P(U)$ of the presheaf P on an open subset U produce sections $\underline{s} : U \to \underline{P}$ of π, defined by $\underline{s}(x) = s_x$, and we can define a new presheaf P^* by taking $P^*(U)$ as the group of those sections

$\sigma : U \to \underline{P}$ of π such that for every point $x \in U$ there is an open neighbourhood $V \subset U$ of x which satisfies $\sigma_{|V} = \underline{s}$ for some $s \in P(V)$

That is, P^* is the presheaf of all sections that locally coincide with sections of P. It can be described in another way by the following construction.

Definition 6.13. The set $\underline{P}$, endowed with the topology whose base of open subsets consists of the sets $s(U)$ for U open in X and $s \in P(U)$, is called the etale space of the presheaf P.

Exercise 6.14. (1) Show that $\pi : \underline{P} \to X$ is a local homeomorphism, i.e., every point $u \in \underline{P}$ has an open neighbourhood U such that $\pi : U \to \pi(U)$ is a homeomorphism.

(2) Show that for every open set $U \subset X$ and every $s \in P(U)$, the section $\underline{s} : U \to \underline{P}$ is continuous.

(3) Prove that P^* is the sheaf of continuous sections of $\pi : \underline{P} \to X$.

(4) Prove that for all $x \in X$ the stalks of P and P^* at x are isomorphic.

(5) Show that there is a presheaf morphism $\phi : P \to P^*$.

(6) Show that ϕ is an isomorphism if and only if P is a sheaf.

P^* is called the sheaf associated with the presheaf P. In general, the morphism $\phi : P \to P^*$ is neither injective nor surjective: for instance, the morphism between the constant presheaf G_X and its associated sheaf G_X is injective but nor surjective.

Example 6.15. As a second example we study the sheaf associated with the presheaf B_X^k of exact k-forms on a differentiable manifold X. For any open set U we have an exact sequence of Abelian groups (actually of R-vector spaces)

$$0 \to B_X^k(U) \to Z_X^k(U) \to H_X^k(U) \to 0$$

where H_X^k is the presheaf that with any open set U associates its k-th de Rham cohomology group, $H_X^k(U) = H_{DR}^k(U)$. Now, the open neighbourhoods of any point $x \in X$ which are diffeomorphic to R^n (where n = dim X) are cofinal[3] in the family of all open neighbourhoods of x, so that $\left(H_X^k\right)_x = 0$ by the Poincare lemma. In accordance with

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Example 6.6 this means that $\left(H_X^k\right)^* = 0$, which is tantamount to $\left(B_X^k\right)^* R\, Z_X^k$.

In this case the natural morhism $H_X^k \to \left(H_X^k\right)^*$ is of course surjective but not injective. On the other hand, $B_X^k \to \left(B_X^k\right)^* = Z_X^k$ is injective but not surjective.

[3]Let I be a directed set. A subset J of I is said to be cofinal if for any $i \in I$ there is a $j \in J$ such that $i < j$. By the definition of direct limit we see that, given a directed family of Abelian groups $\{G_i\}_{i\in I}$, if $\{G_j\}_{j\in J}$ is the subfamily indexed by J, then

$$\varinjlim_{i\in I} G_i R \varinjlim_{J\in J} G_j;$$

that is, direct limits can be taken over cofinal subsets of the index set.

Definition 6.16. Given a sheaf F on a topological space X and a subset (not necessarily open) $S \subset X$, the sections of the sheaf F on S are the continuous sections $\sigma : S \to \underline{F}$ of $\pi : \underline{F} \to X$. The group of such sections is denoted by $\Gamma(S, F)$.

Definition 6.17. Let P, Q be presheaves on a topological space X. [4]

(1) The direct sum of P and Q is the presheaf $P \oplus Q$ given, for every open subset $U \subset X$, by $(P \oplus Q)(U) = P(U) \oplus Q(U)$ with the obvious restriction morphisms.

(2) For any open set $U \subset X$, let us denote by Hom $\left(P_{|U}, Q_{|U}\right)$the space of morphisms between the restricted presheaves $P_{|U}$ and $Q_{|U}$; this is an Abelian group in a natural manner. The presheaf of homomorphisms is the presheaf Hom (P, Q) given by Hom $(P, Q)(U) =$ Hom $\left(P_{|U}, Q_{|U}\right)$ the natural restriction morphisms.

(6) The tensor product of P and Q is the presheaf $(P \otimes Q)(U) = P(U) \otimes Q(U)$

If F and G are sheaves, then the presheaves $F \oplus G$ and $Hom(F, G)$ are sheaves. On the contrary, the tensor product of F and G previously defined may not be a sheaf. Indeed one defines the tensor product of the sheaves F and G as the sheaf associated with the presheaf

$U \to F(U) \otimes G(U)$. It should be noticed that in general $Hom(F,G)(U) \not\cong Hom(F(U),G(U))$ and $Hom(F,G)_x \not\cong Hom(F_x,G_x)$.

6.3 DIRECT AND INVERSE IMAGES OF PRE SHEAVES AND SHEAVES

Here we study the behaviour of presheaves and sheaves under change of base space. Let $f : X \to Y$ be a continuous map.

Definition 6.18. The direct image by f of a presheaf P on X is the presheaf $f_* P$ on Y defined by $(f_* P)(V) = P(f^{-1}(V))$ for every open subset $V \subset Y$. If F is a sheaf on X, then $f_* F$ turns out to be a sheaf.

Let P be a presheaf on Y.

Definition 6.19. The inverse image of P by f is the presheaf on X defined by

$$U \to \varinjlim_{U \subset f^{-1}(V)} P(V).$$

The inverse image sheaf of a sheaf F on Y is the sheaf $f^{-1}F$ associated with the inverse image presheaf of F.

The stalk of the inverse image presheaf at a point $x \in X$ is isomorphic to $P_{f(x)}$. It follows that if $0 \to F' \to F \to F'' \to 0$ is an exact sequence of sheaves on Y , the induced sequence

$$0 \to f^{-1}F' \to f^{-1}F \to f^{-1}F'' \to 0$$

[4]Since we are dealing with Abelian groups, i.e. with $\mathbb{Z}$ -modules, the Hom modules and tensor products are taken over $\mathbb{Z}$. of sheaves on X, is also exact (that is, the inverse image factor for sheaves of Abelian groups is exact).

The etale space $\underline{f^{-1}F}$ of the inverse image sheaf is the fibred product [5]$Y \times_X \underline{F}$. It follows easily that the inverse image of the constant sheaf G_X on X with stalk G is the constant sheaf G_Y with stalk $G, f^{-1}G_X = G_Y$.

Notes

6.4 COHOMOLOGY OF SHEAVES

We wish now to describe a cohomology theory which associates cohomology groups to a sheaf on a topological space X.

6.4.1. Cech cohomology.

We start by considering a presheaf P on X and an open cover U of X. We assume that U is labelled by a totally ordered set I, and define

$$U_{i_0 .....i_p} = U_{i_0} \cap .... \cap U_{i_p}.$$

We define the Cech complex of U with coefficients in P as the complex whose p-th term is the Abelian group

$$\breve{C}^p(U,P) = \prod_{i_0 < ...... < i_p} P\left(U_{i_0 ......i_p}\right).$$

Thus a p-cochain α is a collection $\left\{\alpha_{i_0 ....i_p}\right\}$ of sections of P, each one belonging to the space of sections over the intersection of $p+1$ open sets in U. Since the indexes of the open sets are taken in strictly increasing order, each intersection is counted only once.

The Cech differential $\delta : \breve{C}^p(U,P) \to \breve{C}^{p+1}(U,P)$ is defined as follows: if $\alpha = \left\{\alpha_{i_0 ......i_p}\right\} \in \breve{C}^p(U,P)$, then

$$\left\{(\delta_\alpha)_{i_0 .....i_{p+1}}\right\} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 ....\hat{i}_k .....i_{p+1} | U_{i_0 ....i_{p+1}}}$$

Here a caret denotes omission of the index. For instance, if $p = 0$ we have $\alpha = \{\alpha_i\}$ and

(6.2) $$(\delta\alpha)_{ik} = \alpha_k \left|U_i \cap U_k - \alpha_i\right| U_i \cap U_k.$$

It is an easy exercise to check that $\delta^2 = 0$. Thus we obtain a cohomology theory. We denote the corresponding cohomology groups by $\breve{H}^k(U,P)$.

Lemma 6.1. If F is a sheaf, one has an isomorphism $\breve{H}^0(U,F) \cong F(X)$

Proof. We have $\breve{H}^0(U,F) =$ ker $\delta : \breve{C}^0(U,P) \to \breve{C}^1(U,P)$. So if $\alpha \in \breve{H}^0(U,F)$ by

(6.2) we see that

$$\alpha_{k|U_i \cap U_k} = \alpha_{i|U_i \cap U_k}.$$

By the second sheaf axiom this implies that there is a global section $\alpha \in F(X)$ such that $\alpha_{|U_i} = \alpha_i$. This yields a morphism $\breve{H}^0(U,F) \to F(X)$, which is evidently surjective and is injective because of the first sheaf axiom.

Example 6.2. We consider an open cover U of the circle S^1 formed by three sets which intersect only pairwise. We compute the Cech cohomology of U with coefficients in the constant sheaf R We have $C^0(U,R) = C^1(U,R) = R \oplus R \oplus R, C^k(U,R) = 0$ for $k > 1$ because there are no triple intersections. The only nonzero differential $d_0 : C^0(U,R) \to C^1(U,R)$ is given by

$$d_0(x_0, x_1, x_2) = (x_1 - x_2, x_2 - x_0, x_0 - x_1).$$

Hence

$$\breve{H}^0(U,R) = \ker d_0 R$$

$$\breve{H}^1(U,R) = C^1(U,R) / \operatorname{Im} d_0 R.$$

It is possible to define Cech cohomology groups depending only on the pair (X,F), and not on a cover, by letting

$$\breve{H}^k(X,F) = \varinjlim_{u} \check{H}^k(U,F)$$

The direct limit is taken over a cofinal subset of the directed set of all covers of X (the order is of course the refinement of covers: a cover $U = \{V_j\}_{j \in J}$ is a refinement of U if there is a map $f : I \to J$ such that $V_{f(i)} \subset U_i$ for every $i \in I$). The order must be fixed at the outset, since a cover may be regarded as a refinement of another in many ways.

As different cofinal families give rise to the same inductive limit, the groups $\breve{H}^k(X,F)$ are well defined.

6.4.2 Fine sheaves:

Cech cohomology is well-behaved when the base space X is paracompact. (It is indeed the bad behaviour of Cech cohomology on non-paracompact spaces which motivated the introduction of another cohomology theory for sheaves, usually called sheaf cohomology; cf. [6].) In this and in the following sections we consider some properties of Cech cohomology that hold in that case.

Definition 6.3. A sheaf of rings R on a topological space X is fine if, for any locally finite coper cover $U = \{U_i\}_{i \in I}$ of X^6, there is a family $\{s_i\}_{i \in I}$ of global sections of R such that:

(1) $\sum_{i \in I} s_i = 1$;

(2) for every $i \in I$ there is a closed subset $S_i \subset U_i$ such that $(s_i)_x = 0$ whenever $x \notin S_i$.

[6]We recall that an oper cover U is locally finite if every point in X has an open neighbourhood which intersects only a finite number of elements of U. It is possible to show that whenever X is paracompact, any open cover has a locally finite refinement [17].

The family $\{s_i\}$ is called a partition of unity subordinated to the cover U. For instance, the sheaf of continuous functions on a paracompact topological space as well as the sheaf of smooth functions on a differentiable manifold are fine, while sheaves of complex or real analytic functions are not.

Definition 6.4. A sheaf F of Abelian groups on a topological space X is said to be acyclic if $\check{H}^k(X, F) = 0$ for $k > 0$.

Proposition 6.5. Let R be a _ne sheaf of rings on a paracompact space X. Every sheaf M of R-modules is acyclic.

Proof. Let $U = \{U_i\}_{i \in I}$ be a locally finite open cover of X, and let $\{p_i\}$ be a partition of unity of R subordinated to U. For any $\alpha \in \check{C}^q(U, M)$ with $q > 0$ we set

$$(K\alpha)_{i_0 \ldots\ldots i_{q-1}} = \sum_{\substack{j \in I \\ j < i_0}} p_j q_{j i_0 \ldots\ldots i_{q-1}} - \sum_{\substack{j \in I \\ i_0 < j < i_1}} p_j a_{i_0 j i_1 \ldots\ldots i_{q-1}} + \ldots\ldots$$

$$= \sum_{k=0}^{q} (-1)^k \sum_{\substack{j \in I \\ i_{k-1} < j < i_k}} p_j a_{i_0 \ldots\ldots i_{k-1} j i_k \ldots\ldots i_{q-1}}.$$

This defines a morphism $K:\hat{C}^k(U,M)\to\hat{C}^{k-1}(U,M)$ such that $\delta K+K\delta=id$ (i.e., K is a homotopy operator); then $\alpha=\delta K\alpha$ if $\delta\alpha=0$, so that $\breve{H}^k(U,M)=0$ for $k>0$.

Since on a paracompact space the locally finite open covers are cofinal in the family of all covers, we can take direct limit on such covers, thus getting $\breve{H}^k(X,M)=0$ for $k>0$.

Example 6.6. Using this result we may recast the proof of the exactness of the Mayer-Vietoris sequence for de Rham cohomology in a slightly different form. Given a differentiable manifold X, let U be the open cover formed by two sets U and V . Since $\breve{C}^2(U,\Omega^k)=0$ (there are no triple intersections) we have an exact sequence

$$0\to\breve{H}^{02}(U,\Omega^k)\to\breve{C}^0(U,\Omega^k)\xrightarrow{\ \delta\ }\breve{C}^1(U,\Omega^k)\to 0.$$

which in principle is exact everywhere but at $C^1(U,\Omega^k)$. However since the sheaves Ω^k are acyclic by Proposition 6.5, one has $\breve{H}^1(U,\Omega^k)$, which means that δ is surjective, and the sequence is exact at that place as well. We have the identifications

$$\breve{H}^0(U,\Omega^k)=\Omega^k(X),\quad \to C^0(U,\Omega^k)=\Omega^k U\oplus\Omega^k(V),\qquad C^1(U,\Omega^k)=\Omega^k(U\cap V)$$

so that we obtain the exactness of the Mayer-Vietoris sequence.

6.4.3. Long exact sequences in Cech Cohomology

We wish to show that when X is paracompact, any exact sequence of sheaves induces a corresponding long exact sequence in Cech cohomology.

Lemma 6.7. Let X be any topological space, and let

(6.3) $$0\to P'\to P\to P''\to 0$$

be an exact sequence of presheaves on X. Then one has a long exact sequence

$$0\to\breve{H}^0(X,P')\to\breve{H}^0(X,P)\to\breve{H}^0(X,P'')\to\breve{H}^1(X,P')\to\dots\dots$$

$$\to\breve{H}^k(X,P')\to\breve{H}^k(X,P)\to\breve{H}^k(X,P'')\to\breve{H}^{k+1}(X,P')\to\dots\dots$$

Proof. For any open cover U the exact sequence (6.3) induces an exact sequence of differential complexes

Notes

$$0 \to \check{C}^{\bullet}\left(U, P'\right) \to \check{C}^{\bullet}\left(U, P\right) \to \check{C}^{\bullet}\left(U, P''\right) \to 0$$

which induces the long cohomology sequence

$$0 \to \check{H}^{0}\left(U, P'\right) \to \check{H}^{0}\left(U, P\right) \to \check{H}^{0}\left(U, P''\right) \to \check{H}^{1}\left(U, P'\right) \to ....$$

$$\to \check{H}^{k}\left(U, P'\right) \to \check{H}^{k}\left(U, P\right) \to \check{H}^{k}\left(U, P''\right) \to \check{H}^{k+1}\left(U, P'\right) \to ....$$

Since the direct limit of a family of exact sequences yields an exact sequence, by taking the direct limit over the open covers of X one obtains the required exact sequence.

LEMMA 6.8. Let X be a paracompact topological space, P a presheaf on X whose associated sheaf is the zero sheaf, let U be an open cover of X, and let $\alpha \in \check{C}^{k}\left(U, P\right)$. There is a refinement W of U such that $\tau\left(\alpha\right) = 0$, where $\tau : \check{C}^{k}\left(U, P\right) \to \check{C}^{k}\left(W, P\right)$ is the morphism induced by restriction.

PROOF. We shall need to use the following fact [5, ?]: given an open cover $U = \{U_i\}_{i \in I}$ of a paracompact space X,[7] there is an open cover $U = \{\upsilon_i\}_{i \in I}$ having the same cardinality of U, such that $\overline{V}_i \subset U_i$.

PROPOSITION 6.9. Let P be a presheaf on a paracompact space X, and let P^{*} be the associated sheaf. For all $k \geq 0$, the natural morphism $\check{H}^{k}\left(X, P\right) \to \check{H}^{k}\left(X, P^{*}\right)$ is an isomorphism.

PROOF. One has an exact sequence of presheaves

$$0 \to Q_1 \to P \to P^{*} \to Q_2 \to 0$$

with

(6.4) $$Q_1^{*} = Q_2^{*} = 0.$$

This gives rise to

(6.5)

$$0 \to Q_1 \to P \to T \to 0, \qquad 0 \to T \to P^{*} \to Q_2 \to 0$$

[7]It is enough that X is normal, however, any paracompact space is normal [17].

Where T is the quotient presheaf P / Q_1, i.e. the presheaf $U \to P\left(U\right) / Q_1\left(U\right)$. By Lemma 6.8 the isomorphisms (6.4) yield

$\breve{H}^k(X,Q_1)=\breve{H}^k(X,Q_2)=0$. Then by taking the long exact sequences of cohomology from the exact sequences (6.5) we obtain the desired isomorphism.

Using these results we may eventually prove that on paracompact spaces one has long exact sequences in Cech cohomology.

THEOREM 6.10. Let

$$0\to F^{'}\to F\to F^{''}\to 0$$

be an exact sequence of sheaves on a paracompact space X. There is a long exact sequence of Cech cohomology groups

$$0\to\breve{H}^0(X,F^{'})\to\breve{H}^0(X,F)\to\breve{H}^0(X,F'')\to\breve{H}^1(X,F^{'})\to\ldots\ldots.$$

$$\to\breve{H}^k(X,F^{'})\to\breve{H}^k(X,F)\to\breve{H}^k(X,F'')\to\breve{H}^{k+1}(X,F^{'})\to\ldots\ldots.$$

PROOF. Let P be the quotient presheaf $F/F^{'}$; then $P^{*}\cong F''$. One has an exact sequence of presheaves

$$0\to F^{'}\to F\to P\to 0$$

By taking the associated long exact sequence in cohomology (cf. Lemma 6.7) and using the isomorphism $\breve{H}^k(X,P)=\breve{H}^k(X,F'')$ one obtains the required exact sequence.

Example 6.11. The long exact sequence in cohomology associated with the exact sequence (6.1) starts with

$$0\to H^0(U,\mathbb{Z})\to H^0(U,O)\to H^0(U,O^{*})\to H^1(\mathrm{U},\mathbb{Z})\to\ldots\ldots.$$

This shows that the obstruction to the sequence (6.1) to be exact as a sequence of presheaves is the first cohomology group with coefficients in Z. Since X, being a manifold, is paracompact and locally Euclidean , the Cech cohomology of $\mathbb{Z}$ coincides with the singular cohomology (see Proposition 6.29); therefore the above mentioned obstruction is the non-simply connectedness of U.

Abstract de Rham theorem. We describe now a very useful way of computing cohomology groups; this result is sometimes called “abstract de Rham theorem." As a particular case it yields one form of the so-called de Rham theorem, which states that the de Rham cohomology of a differentiable manifold and the Cech cohomology of the constant sheaf □ are isomorphic.

Notes

DEFINITION 6.12. Let F be a sheaf of abelian groups on X. A resolution of F is a collection of sheaves of abelian groups $\{L^k\}_{k\in\mathbb{Z}}$ with morphisms $i: F \to L^0, d_k : L^k \to L^{k+1}$ such that the sequence

$$0 \to F \xrightarrow{\ i\ } L^0 \xrightarrow{\ d_0\ } L^1 \xrightarrow{\ d_1\ } ....$$

is exact. If the sheaves $L^\bullet$ are acyclic (fine) the resolution is said to be acyclic (fine).

LEMMA 6.13. If $0 \to F \to L^\bullet$ is a resolution, the morphism $i_X : F(X) \to L^0(X)$ is injective.

Proof. Let Q be the quotient L^0 / F. Then the sequence of sheaves

$$0 \to F \to L^0 \to Q \to 0$$

is exact. By Exercise 6.7, the sequence of abelian groups

$$0 \to F(X) \to L^0(X) \to Q(X)$$

is exact. This implies the claim.

However the sequence of abelian groups

$$0 \to L^0(X) \xrightarrow{\ d_0\ } L^1(X) \xrightarrow{\ d_1\ } ....$$

is not exact. We shall consider its cohomology $H^\bullet(L^\bullet(X), d)$. By the previous Lemma we have $H^0(L^\bullet(X), d) R\, H^0(X, F)$.

THEOREM 6.14. If $0 \to F \to L^\bullet$ is an acyclic resolution there is an isomorphism $\breve{H}^k(X,F) R \breve{H}^k(L^\bullet(X), d)$ for all $k \geq 0$.

Proof. Define $Q^k = \ker d_k : L^k \to L^{k+1}$. The resolution may be split into

$$0 \to F \to L_X^0 \to Q^1 \to 0, \quad 0 \to Q^k \to L^k \to Q^{k+1} \to 0. \quad k \geq 1$$

Since the sheaves L^k are acyclic by taking the long exact sequences of cohomology we obtain a chain of isomorphisms

$$\breve{H}^k(X,F) R\, \breve{H}^{k-1}(X, Q^1) R ......R \breve{H}^1(X, Q^{k-1}) R \frac{\breve{H}^0(X, Q^k)}{\operatorname{Im} \breve{H}^0(X, L^{k-1})}$$

By Exercise 6.7 $\breve{H}^0(X, Q^k) R\, Q^k(X)$ is the kernel of $d_k : L^k(X) \to L^{k+1}$ so that the claim is proved.

Corollary 6.15. (de Rham theorem.) Let X be a differentiable manifold. For all $k \geq 0$ the cohomology groups $H^k_{DR}(X)$ and $\breve{H}^k(X, R)$ are isomorphic.

Proof. Let n = dim X. The sequence

(6.6)

$$0 \to R \to \Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \ldots\ldots\ldots \to \Omega_X^n \to 0$$

(where $\Omega_X^0 \equiv C_X^\infty$) is exact (this is Poincare's lemma). Moreover the sheaves $\Omega_X^\bullet$ are modules over the fine sheaf of rings C_X^∞, hence are acyclic. The claim then follows for the previous theorem.

Corollary 6.16. Let U be a subset of a differentiable manifold X which is diffeomorphic to R^n. Then $H^k(U,R)=0$ for $k>0$.

Soft sheaves. For later use we also introduce and study the notion of soft sheaf. However, we do not give the proofs of most claims, for which the reader is referred to [2, 6, 25]. The contents of this subsection will only be used in Section 5.5.

Definition 6.17. Let F be a sheaf a on a topological space X, and let $U \subset X$ be a closed subset of X. The space $F(U)$ (called “the space of sections of F over U") is defined as

$$F(U) = \varinjlim_{V \supset U} F(V)$$

where the direct limit is taken over all open neighbourhoods V of U.

A consequence of this definition is the existence of a natural restriction morphism $F(X) \to F(U)$.

Definition 6.18. A sheaf F is said to be soft if for every closed subset $U \subset X$ the restriction morphism $F(X) \to F(U)$ is surjective.

Proposition 6.19. If $0 \to F' \to F \to F'' \to 0$ is an exact sequence of soft sheaves on a paracompact space X, for every closed subset $V \subset X$ the sequence of groups

$$0 \to F'(V) \to F(V) \to F''(V) \to 0$$

is exact.

Proof. The proof of Proposition 6.2 can be easily adapted to this situation.

Corollary 6.20. The quotient of two soft sheaves on a paracompact space is soft.

Proof. If $F'' = F / F'$ is the quotient of two soft sheaves, by Proposition 6.19 the restriction morphism $F(X) \to F(V)$ is surjective (where V is any closed subset of X), so that $F''(X) \to F''(V)$ is surjective as well.

Proposition 6.21. Any soft sheaf of rings R on a paracompact space is fine.

Proof. Cf. Lemma II.3.4 in [2].

Proposition 6.22. Every sheaf F on a paracompact space admits soft resolutions.

Proof. Let $S^0(F)$ be the sheaf of discontinuous sections of F (i.e., the sheaf of all sections of the sheaf space F). The sheaf $S^0(F)$ is obviously soft. Now we have an exact sequence $0 \to F \to S^0(F) \to F_1 \to 0$. The sheaf F_1 is not soft in general, but it may embedded into the soft sheaf $S^0(F_1)$, and we have an exact sequence $0 \to F_1 \to S^0(F_1) \to F_2 \to 0$. Upon iteration we have exact sequences

$$0 \to F_1 \xrightarrow{i_k} S^k(F) \xrightarrow{p_k} F_{k+1} \to 0$$

where $S^k(F) = S^0(F_k)$.. One can check that the sequence of sheaves

$$0 \to F \to S^0(F) \xrightarrow{f_0} S^1(F) \xrightarrow{f_1} \ldots\ldots.$$

(where $f_k = i_{k+1} \circ p_k$) is exact.

Proposition 6.23. If F is a sheaf on a paracompact space, the sheaf $S^0(F)$ is acyclic.

Proof. The endomorphism sheaf End $(S^0(F))$ is soft, hence fine by Proposition

6.21. Since $S^0(F)$ is an End $(S^0(F))$-module, it is acyclic.[8]

Proposition 6.24. On a paracompact space soft sheaves are acyclic.

Proof. If F is a soft sheaf, the sequence

$0 \to F(X) \to S^0F(X) \to F_1(X) \to 0$ obtained from

$0 \to F \to S^0F \to F_1 \to 0$ is exact (Proposition 6.19). Since F and

S^0F are soft, so is F_1 by Corollary 6.20, and the sequence

$0 \to F_1(X) \to S^1F(X) \to F_2(X) \to 0$ is also exact. With this procedure we can show that the complex $S^\bullet(F)(X)$ is exact. But since all sheaves $S^\bullet(F)$ are acyclic by the previous Proposition, by the abstract de Rham theorem the claim is proved.

Note that in this way we have shown that for any sheaf F on a paracompact space there is a canonical soft resolution.

Leray's theorem for Cech cohomology. If an open cover U of a topological space X is suitably chosen, the Cech cohomologies $\breve{H}^\bullet(U,F)$ and $\breve{H}^\bullet(X,F)$ are isomorphic. Leray's theorem establishes a sufficient condition for such an isomorphism to hold. Since the cohomology $\breve{H}^\bullet(U,F)$ is in generally much easier to compute, this turns out to be a very useful tool in the computation of Cech cohomology groups.

We say that an open cover $U = \{U_i\}_{i\in I}$ of a topological space X is acyclic for a sheaf F if $\breve{H}^k\left(U_{i_0.....i_p}, F\right) = 0$ for all $k > 0$ and all non void intersections $U_{i_0.....i_p} = U_{i_0}k \cap ...... \cap U_{ip}, i_0.....i_p \in I.$

Theorem 6.25. (Leray's theorem) Let F be a sheaf on a paracompact space X, and let U be an open cover of X which is acyclic for F and is indexed by an ordered set. Then, for all $k \geq 0$, the cohomology groups $\breve{H}^k(U,F)$ and $\breve{H}^k(X,F)$ are isomorphic.

To prove this theorem we need to construct the so-called Cech sheaf complex. For every nonvoid intersection $U_{i_0......i_p}$ let $J_{i_0......i_p} : U_{i_0......i_p} \to X$ be the inclusion. For every p define the sheaf

(6.7)

$$\breve{C}^p(U,F) = \prod_{i_0<.......<i_p} \left(j_{i_0.....i_p}\right)_* F_{|U_{i_0.....i_p}}$$

(every factor $\left(j_{i_0.....i_p}\right)_* F_{|U_{i_0......i_p}}$ is the sheaf F first restricted to $U_{i_0......i_p}$ and the extended by zero to the whole of X). The Cech differential induces sheaf morphisms $\delta : \breve{C}^p(U,F) \to \breve{C}^{p+1}(U,F)$. From the definition, we get isomorphisms

(6.8) $$\breve{C}^p(U,F)(X) \cong \breve{C}^p(U,F).$$

[8]We are cheating a little bit, since the sheaf of rings End $\left(S^0(F)\right)$ is not commutative. However a closer inspection of the proof would show that it works anyways.

i.e., by taking global sections of the Cech sheaf complex we get the Cech cochain group complex. Moreover we have:

Lemma 6.26. For all p and k,

$$\breve{H}^k\left(X, \breve{C}^p(U,F)\right) \cong \prod_{i_0<\dots\dots<i_p} \breve{H}^k\left(U_{i_0\dots\dots i_p}, F\right).$$

Proof. By the definition of the Cech cohomology groups we have

$$\breve{H}^k\left(X, \breve{C}^p(U,F)\right) = \varinjlim_{U} \breve{H}^k\left(U, \breve{C}^p(U,F)\right).$$

where U runs over all open covers of X. The groups $\breve{H}^k\left(X, \breve{C}^p(U,F)\right)$ are the cohomology of the complex $\breve{C}^\bullet\left(U, \breve{C}^p(U,F)\right)$, which may be written as

$$\breve{C}^k\left(U, \breve{C}^p(U,F)\right) = \prod_{l_0<\dots\dots<l_k} \breve{C}^p(U,F)\left(V_{l_0\dots\dots i_k}\right)\Big)$$

$$\prod_{l_0<\dots\dots<l_k} \prod_{l_0<\dots\dots<l_p} F\left(V_{l_0\dots\dots l_k} \cap U_{i_0\dots\dots i_p}\right)$$

$$\breve{C}^k\left(U_{i_0\dots\dots i_k}, F_{|U_{i_0\dots\dots i_p}}\right)$$

Where $U_{i_0\dots\dots i_p}$ is the restriction of the cover U to $U_{i_0\dots\dots i_p}$. This implies the claim.

We may now prove Leray's theorem. As an immediate consequence of the fact that F fulfils the sheaf axioms, the complex $\breve{C}^\bullet(U,F)$ is a resolution of F. Under the hypothesis of Leray's theorem, by Lemma 6.26 this resolution is acyclic. By the abstract de Rham theorem, the cohomology of the global sections of the resolution is isomorphic to the cohomology of F. But, due to the isomorphisms (6.8), the cohomology of the global sections of the resolution is the cohomology $\breve{H}^\bullet(U,F)$.

6.5 GOOD COVERS

By means of Leray's theorem we may reduce the problem of computing the Cech cohomology of a differentiable manifold with coefficients in the constant sheaf $\mathbb{R}$ (which, via de Rham theorem, amounts to computing its de Rham cohomology) to the computation of the cohomology of a cover with coefficients in R; thus a problem which in principle would need the solution of differential equations on topologically nontrivial manifolds is reduced to a simpler problem which only involves
the intersection pattern of the open sets of a cover.

Definition 6.27. A locally finite open cover U of a differentiable manifold is good if all nonempty intersections of its members are diffeomorphic to $\mathbb{R}^n$.

Good covers exist on any differentiable manifold (cf. [19]). Due to Corollary 6.16, good covers are acyclic for the constant sheaf $\mathbb{R}$. We have therefore

Proposition 6.28. For any good cover U of a differentiable manifold X one has isomorphisms

$$\check{H}^k(U,R)\,R\,\check{H}^k(X,R), \qquad k \geq 0.$$

The cover of Example 6.2 was good, so we computed there the de Rham cohomology of the circle S^1.

6.6 COMPARISON WITH OTHER COHOMOLOGIES

In algebraic topology one attaches to a topological space X several cohomologies with coefficients in an abelian group G. Loosely speaking, whenever X is paracompact and locally Euclidean, all these cohomologies coincide with the Cech cohomology of X with coefficients in the constant sheaf G. In particular, we have the following result:

Proposition 6.29. Let X be a paracompact locally Euclidean topological space, and let G be an abelian group. The singular cohomology of X with coefficients in G is isomorphic to the Cech cohomology of X with coefficients in the constant sheaf G.

6.7 SHEAF COHOMOLOGY

Another kind of sheaves which can be introduced is that of flabby sheaves (also called “flasque"). A sheaf F on a topological space X is said to be flabby if for every open subset $U \subset X$ the restriction morphism $F(X) \to F(U)$ is surjective. It is easy to prove that flabby sheaves are soft: if $U \subset X$ is a closed subset, by definition of direct limit, for every $s \in F(U)$ there is an open neighbourhood V of U and a section $s' \in F(V)$ which restricts to s. Since F is flabby, s' can be extended to the whole of X. So on a paracompact space, flabby sheaves are acyclic, and by the abstract de Rham theorem flabby resolution can be used to compute cohomology. We should also notice that the canonical soft resolution $S^\bullet(F)$ we constructed in Section 2.5 is flabby, as one can easily check by the definition itself. We shall then call $S^\bullet(F)$ the canonical flabby resolution of the sheaf F (this is also called the Go dement resolution of F).

One can further pursue this line and use flabby resolutions (for instance, the canonical flabby resolution) to define cohomology. That is, for every sheaf F, its cohomology is by definition the cohomology of the global sections of its canonical flabby resolution (it then turns out that cohomology can be computed with any acyclic resolution). This has the advantage of producing a cohomology theory (called sheaf cohomology) which is bell-behaved (e.g., it has long exact sequences in cohomology) on every topological space, not just on paracompact ones. In this section we briefly outline the basics of this theory; for a more comprehensive treatment the reader may refer to [6, 4, 2], or to [23] where a different and more general approach to sheaf cohomology (using injective resolutions) is pursued; also the original paper by Grothendieck [9] can be fruitfully read. It follows from our treatment that on a paracompact topological space the sheaf and Cech cohomology coincide, but in general they do not. In the next chapter we shall establish the relation between the two cohomologies in terms of a spectral sequence (cf. also

[12], especially the exercise section, for a discussion of the comparison between the two cohomologies).

Definition 6.1. If F is a sheaf on a topological space X, its sheaf cohomology groups are defined as

$$H^i(X,F) = H^i\left(S^\bullet(F)(X)\right)$$

for $i \geq 0$

The following two results are basic for this construction. Here X is any topological space.

Proposition 6.2. If $0 \to F' \to F \to F'' \to 0$ is an exact sequence of sheaves, with F' flabby, for any open set $U \subset X$ the sequence of abelian groups

$$0 \to F'(U) \to F(U) \to F''(U) \to 0$$

is exact (namely, the sequence is exact as a sequence of presheaves).

Proof. Let $U \subset X$ and $s'' \in F''(U)$. We need to show the existence of $s \in F(U)$ such that $p(x) = s''$ under the map $p : F \to F''$. Let I be the set of all pairs (W, s), where $W \subset U$ is open, and $s \in F(W)$ represents s'' on W (i.e., $p(s) = s''_{|W}$). The set I is nonempty since the morphism p is surjective in the sense of sheaves. The set I may be given a partial ordering “by extension", i.e., $(W,s) < (W',t)$ if $W \subset W'$ and $s = t_{|W}$. The set has an upper bound (the union of all its elements) and then by Zorn's lemma it has a maximal element $(\overline{W},\overline{s})$. If $x \in U \setminus \overline{W}$ there is a neighbourhood V of x and

a section $t \in F(V)$ which represents s'' in V . Over the intersection $V \cap \overline{W}$ the section $\overline{s} - t$ lies in F' and since F' is flabby it may be extended to V . We can then modify t so that $\overline{s} - t$ in $V \cap \overline{W}$, which contradicts the fact that $(\overline{W},\overline{s})$ is maximal. Then such a x cannot exist, and $\overline{W} = U$.

Corollary 6.6. The quotient of two flabby sheaves is flabby.

Proof. If we have the sequence $0 \to F' \to F \to F'' \to 0$, with F' and F flabby, we may apply the previous Lemma. If $s'' \in F''(U)$ there exists

$s \in F(U)$ such that $p(s) = s''$. Since F is flabby, s extends to a section t of F on X, and then $p(t)$ extends s''.

Corollary 6.4. If

$$0 \to L^0 \to L^1 \to .....$$

is an exact sequence of flabby sheaves, for every open set $U \subset X$ the sequence of abelian groups is exact.

$$0 \to L^0(U) \to L^1(U) \to .....$$

Corollary 6.5. Flabby sheaves are acyclic with respect to sheaf cohomology, i.e., $H^p(X,F) = 0$ for all $p > 0$ if F is a flabby sheaf.

Proof. Apply the previous corollary to the canonical flabby resolution of F.

Corollary 6.6. Flabby sheaves are acyclic with respect to Cech cohomology, i.e., $\breve{H}^p(U,F) = 0$ for every open cover U of X and for all $p > 0$ if F is a flabby sheaf.

Proof. Since F is flabby, the sheaves $\breve{C}^p(U,F)$ defined in Eq. (6.7) are flabby as well. By Corollary 6.4 the sequence

$$\breve{C}^p(U,F)(X) \xrightarrow{\delta} \breve{C}^1(U,F)(X) \xrightarrow{\delta} .....$$

is exact. Since $\breve{C}^p(U,F)(X) = \breve{C}^p(U,F)$, this implies that the Cech complex $\breve{C}^\bullet(U,F)$ is exact.

As a further consequence, we have the isomorphism between Cech and sheaf cohomology on a paracompact space.

Corollary 6.7. For any sheaf F on a paracompact space X, the Cech cohomology $\breve{H}^\bullet(X,F)$ and the sheaf cohomology $H^\bullet(X,F)$ are isomorphic.

Proof. By the previous Corollary, the canonical flabby resolution of F is acyclic for the Cech cohomology, so that the abstract de Rham theorem implies the claim.

We want to show that sheaf cohomology is well behaved with respect to exact sequences of sheaves on any topological space. Let us denote by $Sh/X, K(Sh/X)$ and $K(Ab)$ the categories of sheaves (of abelian groups) on X, of complexes of sheaves on X, and of complexes of abelian groups, respectively. The canonical flabby resolution allows one to define two functors:

$$F_1 : Sh/X \to K(Sh/X)$$

$$F \mapsto S^\bullet(F)$$

$$F_2 : Sh/X \to K(Ab)$$

$$F \mapsto S^\bullet(F)(X)$$

Proposition 6.8. The two functors F_1, F_2 are exact (i.e., they map exact sequences to exact sequences).

Proof. If

(6.9) $$0 \to F' \to F \to F'' \to 0$$

is an exact sequence of sheaves we have for every x 2 X an exact sequence

$$0 \to \prod_{x\in H} F_x^{'} \to \prod_{x\in X} F_x \to \prod_{x\in X} F_x^{''} \to 0$$

so that the sequence of complexes of sheaves

$$0 \to s^\bullet(F') \to s^\bullet(F) \to s^\bullet(F'') \to 0$$

induced by (6.9) is exact. This proves that F_1 is exact. Moreover, by Proposition 6.2 the sequence

(6.10) $$0 \to s^\bullet(F')(X) \to s^\bullet(F)(X) \to s^\bullet(F'')(X) \to 0$$

is exact as well, so that F_2 is exact.

Corollary 6.9. If $0 \to F' \to F \to F'' \to 0$ is an exact sequence of sheaves, there is a long exact sequence of cohomology

(6.11)

$$0 \to H^0(X,F^{'}) \to H^0(X,F) \to H^0(X,F'') \to H^1(X,F^{'}) \to ....$$

$$\to H^k(X,F^{'}) \to H^k(X,F) \to H^k(X,F'') \to H^{k+1}(X,F^{'}) \to ....$$

Proof. The long exact sequence of cohomology associated with the exact sequence of complexes of abelian groups (6.10) is just the sequence (6.11).

An immediate consequence of this result is that the proof of the abstract de Rham theorem for the Cech cohomology on a paracompact space may be applied to provide a proof of the same theorem for sheaf cohomology on any space; thus, on any topological space, the sheaf cohomology of a sheaf F is isomorphic to the cohomology of the complex of global sections of a resolution of F which is acyclic for the sheaf cohomology.

Check Your Progress

1. Prove: If $0 \to F \to L^{\bullet}$ is an acyclic resolution there is an isomorphism $\breve{H}^k(X,F) R\breve{H}^k(L^{\bullet}(X),d)$ for all $k \geq 0$

.____

2. Explain about Cohomology of sheaves

3. Explain about Sheaf Cohomology

6.8 LET US SUM UP

A pre sheaf of Abelian groups on X is a rule[1] P which assigns an Abelian group P(U) to each open subset U of X and a morphism (called restriction map) $\varphi_{U,V} : P(U) \to P(V)$ to each pair $V \subset U$ of open subsets, so as to verify the following requirements:

(1) $P(\emptyset) = \{0\}$;

(2) $\varphi_{U,V}$ is the identity map;

(3) if $W \subset V \subset U$ are open sets, then $\varphi_{U,W} = \varphi_{V,W} \circ \varphi_{U,V}$.

Given a sheaf F on a topological space X and a subset (not necessarily open) $S \subset X$, the sections of the sheaf F on S are the continuous sections $\sigma : S \to \underline{F}$ of $\pi : \underline{F} \to X$. The group of such sections is denoted by $\Gamma(S,F)$.

The inverse image of P by f is the presheaf on X defined by

$$U \to \varinjlim_{U \subset f^{-1}(V)} P(V).$$

The inverse image sheaf of a sheaf F on Y is the sheaf $f^{-1}F$ associated with the inverse image presheaf of F.

6.9 KEY WORDS

Direct and inverse images of pre sheaves and sheaves

Cohomology of sheaves

Sheaf Cohomology

Fine sheaves

6.10 QUESTIONS FOR REVIEW

1. Explain about direct and inverse images of pre sheaves and sheaves
2. Explain about Cohomology of sheaves
3. Explain about long exact sequences in cesh Cohomology

6.11 SUGGESTIVE READINGS AND REFERENCES

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6.12 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. See sub section 6.7
2. See section 6.6
3. See section 6.7

UNIT-7 BORDISM SPECTRA, AND GENERALIZED HOMOLOGY

STRUCTURE

7.0 OBJECTIVE

In this unit we will learn understand about Framed bordism and homotopy groups of spheres, Suspension and the frudenthal theorem, Stable tangential framings.

7.1 INTRODUCTION

This chapter contains a mixture of algebraic and differential topology and serves as an introduction to generalized homology theories. We will give

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a precise definition of a generalized homology theory later, but in the mean- time you should think of a generalized homology theory as a functor from pairs of spaces to graded abelian groups (or graded R-modules) satisfying all Eilenberg–Steenrod axioms but the dimension axiom.

The material in this chapter will draw on the basic notions and theo-rems of differential topology, and you should re-familiarize yourself with the notion of smooth maps between smooth manifolds, sub manifolds, tangent bundles, orientation of a vector bundle, the normal bundle of a sub manifold, the Sard theorem, transversality and the tubular neighborhood theorem. One of the projects for this chapter is to prepare a lecture on these topics.

7.2 FRAMED BORDISM AND HOMOTOPY GROUPS OF SPHERES

Pontrjagin and Thom in the 1950's noted that in many situations there is a one-to-one correspondence between problems in geometric topology (= manifold theory) and problems in algebraic topology. Usually the algebraic problem is more tractable, and its solution leads to geometric consequences. In this section we discuss the quintessential example of thiscorrespondence; are ferencei sthelastsect ionof Milnor'sbeautifullittlebo ok[27].

We start with an informal discussion of the passage from geometric topology to algebraic topology.

Definition 7.1. A framing of a submanifold V^{k-n} s of a closed manifold M^k is a embedding ϕ of $V \times R^n$ in M so that $\phi(p,0) = p$ for all $p \in V$. If (W^{k+1-n}, ψ) is a framed submanifold of $M \times I$, then the two framed submanifolds of M given by intersecting W with $M \times \{0\}$ and $M \times \{1\}$ are framed bordant. Let $\Omega^{fr}_{k-n,M}$ be the set of framed bordism classes of $(k-n)$ – dimensional framed submanifolds of M.

A framed submanifold defines a collapse map $M \to S^n = R^n \cup \{\infty\}$ by sending $\phi(p,\upsilon)$ to υ and all points outside the image of 0 is V. A framed bordism gives a homotopy of the two collapse maps. A framed bordism from a framed submanifold to the empty set is a null-bordism. In the special form a framed submanifold V^{k-n} of S^k, a null-bordism is given by an extension to a framed submanifold W^{k+1-n} of D^{k+1}.

Theorem 7.2. The collapse map induces a bijection $\Omega^{fr}_{k-n,M} \to [M, S^n]$.

This method of translating between bordism and homotopy sets is called the Pontrjagin-Thom construction.

Here are some examples (without proof) to help your geometric insight. A (framed) point in a S^k gives a map $S^k \to S^K$ which generates $\pi_K S^k \cong Z.$ Any framed circle is S^2 is null-bordant, for example the equator with the obvious framing is the boundary of the 2-disk in the 3-ball. However, a framed S^1 in S^3 so that the circle $\phi(S^1 \times \{(1,0)\})$ links the S^1 with linking number 1 represents the generator of $\pi_3(S^2) \cong Z.$ (Can you reinterpret this in terms of the Hopf map? Why can't one see the complexities of knot theory in framed bordism?) Now S^3 is naturally farmed in S^4, S^4 in S^5, etc. So we can suspend the linking number 1 framing of S^1 in S^3 to get a framing of S^1 in S^{k+1} for $k > 2$. this represents the generator of $\pi_{k+1} S^k \cong Z_2$.

More generally, one can produce examples of framed manifolds by twisting and suspending. If (V^{k-n}, ϕ) is a framed submanifold of M^k and $\alpha : V \to O(n)$, then the twist is the framed submanifold $(V, \phi.\alpha)$ where $\phi.\alpha(p,\upsilon) = \phi(p, \alpha(p)\upsilon)$. The framed bordism class depends only on (V,ϕ) and the homotopy class of α. (See Exercise 132 below for more on this construction.) Next if (V^{k-n},ϕ) is a framed submanifold of S^k, then the suspension of (V^{k-n},ϕ) is a framed submanifold $(V^{k-n}, S\phi)$ of S^{k+1} is defined using the obvious framing of

S^k in S^{k+1}, with $S^k \times R_{>0}$ mapping to the upper hemisphere of S^{k+1}.

Then the generator of $\pi_3(S^2)$ mentioned earlier can be described by first suspending the inclusion of a framed circle in the 2-sphere, and then twisting by the inclusion of the circle in O(2).

To prove Theorem 7.2 we first want to reinterpret $\Omega^{fr}_{k-n,M}$ in terms of normal framings. The key observation is that a farmed submanifold determines n linearly independent normal vector fields on M.

Definition 7.3.

1. A trivialization of a vector bundle $p : E \to B$ with fiber R^n is a collection $\{\sigma_i : B \to E\}_{i=1}^{n}$ of sections which form a basis point wise. Thus $\{\sigma_1(b),\ldots,\sigma_n(b)\}$ is linearly independent and spans the fiber E_b for each $b \in B$.

Equivalently, a trivialization is a specific bundle isomorphism $E \cong B \times R^n$. A trivialization is also the same as a choice of section of the associated principle frame bundle.

2. A farming of a vector bundle is a homotopy class of trivializations, where two trivializations are called homotopic if there is a continuous 1-parameter family of trivializations joining them. In terms of the associated frame bundle this says the two sections are homotopic in the space of sections of the frame bundle.

A section of a normal bundle is called a normal vector field.

Definition 7.4. A normal framing of a submanifold V of m is a homotopy class of trivializations of the normal bundle $\upsilon(V \mapsto M)$. If W is a normally framed submanifold of $M \times I$, then the two normally framed submanifolds of m given by intersecting W with $M \times \{0\}$ and $M \times \{1\}$ are normally framed bordant. (You should convience yourself that restriction of $\upsilon(W \mapsto M \times I)$ to $V_0 = (M \times \{0\}) \cap W$ is canonically identified with $\upsilon(V_0 \mapsto M)$).

Exercise : Show that a framed submanifold (V,ϕ) of M determines a normal framing of V in M. use notation from differential geometry and denote the standard coordinate vector fields on R^n by $\{\partial/\partial x_1,\ldots,\partial/\partial x_n\}$.

Exercise : Define a map from the set of bordism classes of $(k-n)-$ dimensional framed submanifolds of M to the set of bordism classes of $(k-n)-$ dimensional normally framed submanifolds of M and show it is a bijection. (The existence part of the tubular neighborhood theorem will show the map is surjective, while the uniqueness part will show the map is injective.)

Henceforth we let $\Omega^{fr}_{k-n,M}$ denote both the bordism classes of framed submanifolds and the bordism classes of normally framed submanifolds of M.

Proof of Theorem 7.2. To define an inverse

$$d:[M,S^n]\to\Omega^{fr}_{k-n,M}$$

to the collapse map

$$c:\Omega^{fr}_{k-n,M}\to[M,S^n]$$

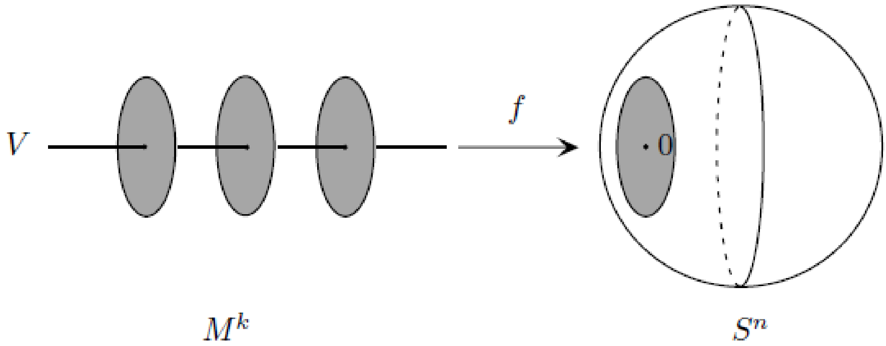
One must use differential topology; in fact, this was the original motivation for the development of transversality.

Any element of $[M,S^n]$ can be represented by a map $f:M\to S^n=R^n\cup\{\infty\}$, which is smooth in a neighborhood of $f^{-1}(0)$ and transverse to 0 (i.e. 0 is a regular value). Thus:

1. The inverse image $f^{-1}(0)=V$ is a smooth submanifold of M^k of codimension n (i.e. of dimension k-n), and
2. The differential of f identifies the normal bundle of V in M^k with the pullback of the normal bundle of $0\in S^n$ via f. More precisely, the differential of f, $df:TM^k\to TS^n$ restricts to $TM^k\,|\,V$ and factors through the quotient $\upsilon(V\mapsto M^k)$ to give a map of vector bundles

$$\begin{array}{ccc} \nu(V \hookrightarrow M^k) & \xrightarrow{df} & \nu(0 \hookrightarrow S^n) \\ \downarrow & & \downarrow \\ V & \longrightarrow & 0 \end{array}$$

Which is an isomorphism in each fiber.



Since the normal bundle of 0 in $R^n \cup \{\infty\}$ is naturally framed by the standard basic, the second assertion above implies that the normal bundle of $V \, in \, M^k$ is also framed, i.e. there is a bundle isomorphism

$$\begin{array}{ccc} \nu(V \hookrightarrow M^k) & \xrightarrow{\cong} & V \times \mathbf{R}^n \\ & \searrow \quad \swarrow & \\ & V & \end{array}$$

The map d is defined by sending $[f] \, to \, f^{-1}(0)$ with the above framing. To see that d is well-defined, consider a homotopy

$$F : M \times I \to S^n.$$

Where $F|_{M \times \{0,1\}}$ is transverse to $0 \in S^n$. Consider the "trace of F"

$$\begin{aligned} \hat{F} : M \times I &\to S^n \times I \\ (m,t) &\mapsto (F(m,t),t), \end{aligned}$$

Which has the advantage that it takes boundary points to boundary points. The (relative) transversality approxiamation theorem says that $\hat{F}$ is homo-topic $(\text{rel} M \times \{0,1\})$ to a map transverse to $0 \times I$. The inverse image of $0 \times I$ equipped with an appropriate normal framing gives a normally framed bordism between $F|_{M \times \{0\}}^{-1}(0)$ and $F|_{M \times \{1\}}^{-1}(0)$.

Our final task is to show that c and d are mutual inverses. It is easy to see that $c \circ d$ is the identity takes some work. First represent an element of

$\left[M, S^n\right]$ by a map f transverse to $0 \in R^n \cup \{\infty\} = S^n$. It seems plausible that the collapse map associated to $V = f^{-1}(0)$ with the normal framing induced by df is homotopic to f, but there are technical details. Here goes. Let $\upsilon = \upsilon(V \mapsto M)$, let $g : \upsilon \to M$ be a tubular neighborhood of V, assume υ has a metric, and let $D = g(D(\upsilon))$ correspond to the disk bundle. Define $\Phi : \upsilon \to R^n$ by $\Phi(x) = \lim_{t \to 0} t^{-1} f(g(tx))$. Then $\Phi(x)$ is the velocity vector of a curve, and by the chain rule Φ is the composite of the identification of υ with $\upsilon(V \mapsto \upsilon)$ and $df \circ dg$. In particular Φ gives an isomorphism from each fiber of υ to R^n.

There is a homotopy $f_t : D \to R^n \cup \{\infty\}$ for $-1 \le t \le 1$ given by

$f_t(g(x)) = \{ \longrightarrow \Phi(x) \quad \text{if } -1 \le t \le 0,$

We now have a map

$\partial D \times [-1,1] \cup (M - \text{Int} D) \times \{1\} \cup (M - \text{Int} D) \times \{-1\} \to S^n - \{0\}$ Defined by f_t on the first piece, by f on the second piece, and by the constant map at infinity on the third piece. This extends to a map

$(M - \text{Int} D) \times [-1,1] \to S^n - \{0\}$ by the Tietze extension theorem.

We can then paste back in f_t to get a homotopy

$$F : M \times [-1,1] \to S^n$$

From our original f to a map h so that

$h^{-1} R^n = \text{Int} D \cong V \times R^n$

Where the diffeomorphism $\cong$ is defined by mapping to V by using the original tubular neighborhood and by mapping to R^n by h. Thus $f \simeq h$ where h is in the image of c. It follows the c is surjective and thus that c and d are mutual inverses.

In reading the above proof you need either a fir amount of technical skill to fill in the details or you need to be credulous. For an alternate approach see [27, Chapter 7].

For a real vector bundle over a point, i.e. a vector space, a framing is the same as a choice of orientation of the vector space, since $GL(n, R)$ has two path components. Thus a normal framing of $V \subset S^k$ induces an

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orientation on the normal bundle $\upsilon\left(V \to S^k\right)$. (See Section 10.7 for more information about orientation.)

Exercise: Let V be a normally framed submanifold of a manifold M. Show that an orientation of M induces an orientation of V. (Hint: Consider the isomorphism $TV \oplus \upsilon = TM/_V$.)

Theorem 7.5 (Hopf degree theorem). Let M^k be a connected, closed, smooth manifold.

1. If M^k is orientable, then two maps $M^k \to S^k$ are homotopic if and only if they have the same degree.

2. If M^k is nonorientable, then two maps $M^k \to S^k$ are homotopic if and only if they have the same degree mod 2.

Exercise : Prove the Hopf degree theorem in two ways: obstruction theory and theory and framed bordism.

The function $\pi_k S^n \to \left[S^k, S^n\right]$ obtained by forgetting base points is a bijection. For n >1 this follows from the fact that S^n is simply connected and so vacuously the fundamental group acts trivially. For n=1 this is still true because $\pi_k S^1$ is trivial for k > 1 and abelian for k=1.

The result that $\pi_n S^n \cong Z$ is a nontrivial result in algebraic topology; it is cool that this can be proven using differential topology.

Exercise. We only showed that the isomorphism of theorem 7.2 is a bijection of sets. However, since $\pi_n S^n$ is an abelian group, the framed bordism classes inherit an abelian group structure. Prove that this group structure on framed bordism is given by taking the disjoint union:

$$[V_0]+[V_1] : V_0 II V_1 \subset S^k \neq S^k \cong S^k$$

With negatives given by changing the orientation of the framing (e.g.replacing to first vector field in the framing by its negative)

$$-[V_0]=[-V_1].$$

We wil generalize Theorem 7.2 by considering the effect of the suspension map $S : \pi_k S^n \to \pi_{k+1} S^{n+1}$ and eventually passing to the limit $\lim_{\ell\to\infty} \pi_{k+\ell} S^{n+1}$. This has the effect of eliminating the thorny embedding questions of submanifolds in S^k; in the end we will be able work with abstract framed manifolds V without reference to an embedding of V in some sphere.

Exercise : (The J-homorphism) Let $V^{k-n} \subset M^k$ be a non-empty normally framed manifold. Used Twisting to define a function

$$J : \left[V^{k-n}, O(n)\right] \to \left[M^k, S^n\right].$$

Now let V be the equatorial $S^{k-n} \subseteq S^k$ with the canonical framing coming from the inclusions $S^{k-n} \subseteq S^{k-n+1} \subset \ldots \subset S^k$, and show that the function

$$J : \pi_{k-n}\left(O(n)\right) \to \pi_k S^n$$

is a homomorphism provided k > n. It is called the J-homomorphism and can be used to construct interesting elements in $\pi_k S^n$.

Draw and explicit picture of a framed circle in $R^3 = S^3 - \{\infty\}$ representing $J(t)$ where $t \in \pi_1 O(2) = Z$ is the generator.

7.3 SUSPENSION AND THE FREUDENTHAL THEOREM

Recall that (reduced) suspension of a space $X \in K_*$ with no degenerate base point is the space

$$SX = X \times I / \sim$$

Where the subspace $(x_0 \times I) \cup \left(X \times \{0,1\}\right)$ is collapsed to a point. This construction is functorial with respect to based maps $f : X \to Y$. In particular, the suspension defines a function

$$S : [X, Y]_0 \to [SX, SY]_0 .$$

Notes

By proposition 6.35, $SS^k = S^{k+1}$, so that when $X = S^k$, the suspension defines a function, in fact a homorphism

$$S : \pi_k(Y) \to \pi_{k+1}(SY)$$

For any space Y. Taking Y to be a sphere one obtains

$$S : \pi_k(S^n) \to \pi_{k+1}(S^{n+1}).$$

We next identify $S^k \subset S^{k+1} = SS^k$ as the equator, and similarly $S^n \subset S^{n+1}$, and interpret the above map in terms of framed bordism.

If $f : S^k \to S^n$ is smooth, then the suspension

$Sf : S^{k+1} \to S^{n+1}$

Is smooth away from the base points. If $x \in S^n$ is a regular value different from the base point, and $V = f^{-1}(x)$ is the normally framed submanifold of S^k associated to f, then clearly

$$V = (Sf)^{-1}(x) \subset S^k \subset S^{k+1}.$$

Let us compare normal bundles and normal framings.

$$\begin{aligned} \upsilon(V \to S^{K+1}) &= \upsilon(V \to S^k) \oplus \upsilon(S^k \to S^{k+1}) \mid V \\ &= \upsilon(V \to S^k) \oplus \in_V \end{aligned}$$

Where $\in_V = V \times R =$ trivial line bundle.

Similarly, $\upsilon(x \to S^{n+1}) = \upsilon(x \to S^n) \oplus \in_{\{x\}}$, and the differential of Sf preserves the trivial factor, since, locally (near the equator $S^k \subset S^{k+1}$),

$Sf \cong f \times Id : S^k \times (-\in, \in) \to S^n \times (-\in, \in).$

We have shown the following.

Theorem 7.6. Taking the suspension of a map corresponds, via the Pontrjagin- Thom construction, to the same manifold V, but embedded in the equation $S^k \subset S^{k+1}$, and with normal framing the direct sum of the old normal framing and the trivial 1-dimensional framing.

Now consider the effect of multiple suspensions.

$$S^{\ell} : \pi_k S^n \to \pi_{k+\ell} S^{n+\ell}$$

For each suspension, the effect on the normally framed submanifold V is to replace it by the same manifold embedded in the equator, with the new normal framing $\upsilon_{new} = \upsilon_{old} \oplus \in_V$. Thus after ℓ suspensions, $\upsilon_{new} = \upsilon_{old} \oplus \in_V^{\ell}$.

The following fundamental result is the starting point for the investigation of "stable" phenomena in homotopy theory. We will not give a proof at this time, since a spectral sequence proof is the easiest way to go. The proof is given in Section 10.3.

Theorem 7.7 (Freudenthal suspension theorem). Suppose that X is an $(n-1)-$ connected space $(n \geq 2)$. Then the suspension homomorphism

$$S : \pi_K X \to \pi_{k+1} SX$$

Is an isomorphism if k<2n-1 and an epimorphism if k=2n-1.

The most important case is when $X = S^n$, and here Freudenthal suspension theorem can also be given a differential topology proof using framed bordism and the facts that any j-manofold embeds in S^n for $n \geq 2j+1$, uniquely up to isotopy if $n \geq 2j+2$, and that any embedding of a j-manifold in S^{n+1} is isotopic to an embedding in S^n if $n \geq 2j+1$.

Exercise: Show that for any k-dimensional CW-complex X and for any (n-1)-connected space $y(n \geq 2)$ the suspension map

$$[X, Y]_0 \to [SX, SY]_0$$

Is bijective if $k < 2n-1$ and surjection of $k = 2n-1$. (Hint:Instead consider the map $[X, Y]_0 \to [X, \Omega SY]_0$. Convert the map $Y \to \Omega SY$ to a fibration and apply cross-section obstruction theory as well as the Freudenthal suspension theorem).

For a based space $X, \pi^n X = \left[X, S^n\right]_0$ is called the n-th cohomotopy set. If X is a CW-complex with dim $X < 2n-1$, then Exercise 133 implies that

$\pi^n X$ is a group structure given by suspending and using the suspension coordinate in SX. The reader might ponder the geometric meaning (framed bordism) of the cohomotopy group structure when X is a manifold.

Definition 7.8. The k-th stable homotopy group of a based space X is the limit

$$\pi_k^S X = \lim_{\ell\to\infty} S^\ell X.$$

The stable k-stem is

$$\pi_k^S = \pi_k^S S^0.$$

The computation of the stable k-stem for all k is the holy grail of the field of homotopy theory.

The Hurewicz theorem implies that if X is $(n-1)-$connected, then SX is n-connected, since $\tilde{H}_\ell SX = \tilde{H}_{\ell-1} X = 0$ if $\ell \le n$ and $\pi_1 SX = 0$ if X is path connected. The following corollary follows from this fact and the Freudenthal theorem.

Corollary 7.9. If X is path connected,

$$\pi_k^S X = \pi_{2k}\left(S^k X\right) = \pi_{k+\ell}\left(S^\ell X\right) \qquad \text{for } \ell \ge k.$$

For the stable k-stem,

$$\pi_k^S = \pi_{2k+2}\left(S^{k+2}\right) = \pi_{k+\ell}\left(S^\ell\right) \qquad \text{for } \ell \ge k+2..$$

Recall from Equation(6.3) that $\pi_k\left(O(n-1)\right) \to \pi_k\left(O(n)\right)$, induced by the inclusion $O(n-1) \to O(n)$, is an isomorphism for $k < n-2$, and therefore letting $O = \lim_{n\to\infty} O(n), \pi_k\left(O(n)\right)$ for $k < n-2$. It follows from the definitions that the following diagram commutes

$$\begin{array}{ccc} \pi_k(O(n-1)) & \xrightarrow{J} & \pi_{k+n-1}(S^{n-1}) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle s} \\ \pi_k(O(n)) & \xrightarrow{J} & \pi_{k+n}(S^n) \end{array}$$

With the horizontal maps the J-homomorphisms, the left vertical map induced by the inclusion and the right vertical map the suspension homomorphism. If $k < n-2$, then both vertical maps are isomorphisms, and so one obtains the stable J-homomorphism

$$J : \pi_k(O) \to \pi_k^S.$$

Corollary 7.10. The Pontrjagin-Thom construction defines an isomorphism from π_k^S to the normally framed bordism classes of normally framed k-dimensional closed submanifolds of $J : \pi_k(O) \to S^n$ for any $n \geq 2k+2$.

7.4 STABLE TANGENTIAL FRAMINGS

We wish to remove the restriction that our normally framed manifolds be submanifolds of To this end we need to eliminate the reference to the normal bundle. This turns out to be easy and corresponds to the fact that the normal and tangent bundles of a submanifold of S^n. are inverses in a certain stable sense. Since the tangent bundle is an intrinsic invariant of a smooth manifold, and so is defined independently of any embedding in S^k, this will enable us to replace normal framings with tangential framings. On the homotopy level, however, we will need to take suspensions whendescribing in what way the bundles are inverses. In the end this means that we will obtain an isomorphism between stably tangentially framed bordism classes and stable homotopy groups.

In what follows, $\in^j$ will denote a trivialized j-dimenasional real bundle over a space.

Lemma 7.11. Let $V^k \subset S^n$ be a closed, oriented, normally framed sub-manifold of S^n. Then

1. A normal framing $\gamma : \upsilon\left(V \to S^n\right) \cong \in^{n-k}$ induces a trivialization

$$\overline{\gamma} : TV \oplus \in^{n-k+1} \cong \in^{n+1} .$$

2. A trivialization $\overline{\gamma} : TV \oplus \cong \in^{k+1}$. induces a trivialization

$$\upsilon\left(V \to S^n\right) \oplus \in^{k+1} \cong \in^{n+1} .$$

Proof. The inclusion $S^n \subset R^{n+1}$ has a trivial 1-dimensional normal bundle which can be framed by choosing the outward unit normal as a basic. This shows that the once stabilized tangent bundle of S^n is canonically trivialized

$$TS^n \oplus \in \cong \in^{n+1}$$

Since the tangent bundle of R^{n+1} is canonically trivialized.

There is a canonical decomposition

$$\left(TS^n \oplus \in\right)|_V = \upsilon\left(V \to S^n\right) \oplus TV \oplus \in .$$

Using the trivialization of $TS^n \oplus \Sigma$, one has a canonical isomorphism

$$\Sigma^{n+1} \cong \upsilon\left(V \to S^n\right) \oplus TV \oplus \Sigma.$$

Thus a normal framing $\gamma : \upsilon\left(V \to S^k\right) \cong \Sigma^{n-k}$ induces an isomorphism

$$\Sigma^{n+k} \cong \Sigma^{n-k} \oplus TV \oplus \Sigma,$$

And, conversely a trivialization $\overline{\gamma} : TV \oplus \Sigma \cong \Sigma^{k+1}$ induces an isomorphism

$$\Sigma^{n+k} \cong \upsilon\left(V \to S^n\right) \oplus \Sigma^{k+1}$$

Definition 7.12. A stable (tangential) framing of an k-dimensional manifold V is an equivalence class of trivializations of

$$TV \oplus \Sigma^n$$

Where Σ^n is the trivial bundle $V \times R^n$. Two trivializations

$$t_1 : TV \oplus \Sigma^{n_1} \cong \Sigma^{k+n_1}, t_2 : TV \oplus \Sigma^{n_2} \cong \Sigma^{k+n_2}$$

Are considered equivalent if there exists some N greater than $n_1 \text{ and } n_2$ such than the direct sum trivializations

$$t_1 \oplus : Id : TV \oplus \Sigma^{n_1} \oplus \Sigma^{N-n_1} \cong \Sigma^{k+n_1} \oplus \Sigma^{N-n_1} = \Sigma^{k+N}$$

And

$$t_2 \oplus : IId : TV \oplus \Sigma^{n_2} \oplus \Sigma^{N-n_2} \cong \Sigma^{k+n_2} \oplus \Sigma^{N-n_2} = \Sigma^{k+N}$$

Are homotopic.

Similarlay, a stable normal framing of a submanifold V of S^{ℓ} is an equivalence class of trivializations of $\upsilon\left(V \to S^{\ell}\right) \oplus \Sigma^{n}$ and a stable framing of a bundle η is an equivalence class of trivializations of $\eta \oplus \Sigma^{n}$.

A tangential framing is easier to work with than a normal framing, since one does not need to refer to an embedding $V \subset S^{n}$ to define a tangential framing. However, stable normal framings and stable tangential framings are equivalent; essentially because the tangent bundle of S^{n} is canonically stably framed. Lemma 7.11 generalizes to give the following theorem.

Theorem 7.13. There is a 1-1 correspondence between stable tangential framings and stable normal framing of a manifold V. More precisely:

1. Let $i: V \to S^{n}$ be an embedding. A stable framing of TV determines stable framing of $\upsilon(i)$ and conversely.

2. Let $i_1 : V \to S^{n_1}$ and $i_2 : V \to S^{n_2}$ be embeddings. For n large enough there exists a canonical (up to homotopy) identification

$$\upsilon(i_1) \oplus \Sigma^{n-n_1} \cong \upsilon(i_2) \oplus \Sigma^{n-n_2}.$$

A stable framing of $\upsilon(i_1)$ determines one of $\upsilon(i_2)$ and vice versa.

Proof. 1. The proof of Lemma 7.11 gives a canonical identification

$$\upsilon\left(V \to S^{n}\right) \oplus \Sigma^{\ell} TV \cong \Sigma^{n+\ell}$$

For all $\ell > 0$. Associativity of $\oplus$ shows stable framings of the niraml bundle and tangent bundles coincide.

3. Let $i_1 : V \to S^{n_1}$ and $i_2 : V \to S^{n_2}$ be embeddings. There is a formal proof that stable framings of $\upsilon(i_1)$ and $\upsilon(i_2)$ coincide. Namely, a stable framing of $\upsilon(i_1)$ determines a stable framing of TV by part 1, which in turn determines a stable framing of $\upsilon(i_2)$. However, the full statement of part 2 applies to submanifolds with non-trivial normal bundle, and theorems from differential topology must be used.

Choose n large enough so that any two embeddines of V in S^{n} are isotopic. (Tranversality theorems imply that $n > 2k+1$ suffices.)

The composite $V \xrightarrow{i_1} S^{n_1} \xrightarrow{j_1} S^n$, with $S^{n_1} \xrightarrow{j_1} S^n$ the equatorial embedding, has normal bundle

$$\upsilon(j_1 \circ i_1) = \upsilon(i_1) \oplus \Sigma^{n-n_1}.$$

Similaraly, the composite $V \xrightarrow{i2} S^{n_2} \xrightarrow{j_2} S^n$, has normal bundle

$$\upsilon(j_2 \circ i_2) = \upsilon(i_2) \oplus \Sigma^{n-n_2}.$$

Then $j_2 \circ i_2$ is isotopic to $j_1 \circ i_1$, and the isotropy induces an isomorphism

$$\upsilon(j_2 \circ i_2) \cong \upsilon(j_1 \circ i_1).$$

If $n > 2(k+1)+1,$ then any self-isotopy is isotopic to the constant isotopy, so that the isentificfation $\upsilon(j_2 \circ i_2) \cong \upsilon(j_1 \circ i_1).$ is canonical (up to homotopy.)

Definition 7.14. Two real vector bundles E, F over V are called stably equivalent if there exists non-negative integers I,j so that $E \oplus \Sigma^i$ and $F \oplus \Sigma^i$ are isomorphic.

Since every smooth compact manifold embeds S^n for some n, the second part of Theorem 7.13 has the consequence that the stable normal bundle (i.e. the stable equivalence class of the normal bundle for some embedding) is a well defined invariant of a smooth manifold, independent of the embedding, just as the tangent bundle is. However, something stronger holds. If $\upsilon(i_1)$ and $\upsilon(i_2)$ are normal bundles of two different embeddings of a manifold in a sphere, then not only are $\upsilon(i_1)$ and $\upsilon(i_2)$ stably equivalent, but the stable isomorphism is determined up to homotopy.

Returinig to bordism. We sww that the inclusion $S^n \subset S^{n+1}$ sets up a correspondence between the suspension operation and stabilizing a normal (or equivalently tangential) framing. Consequently Corollary 7.10 can be restated as follows.

Corollary 7.15. The stable k-stem π_k^S is isomorphic to the stably tangentially framed bordism classes of stably tangentially framed k-dimensional smooth, oriented manifolds without boundary.

This statement is more appealing since it refers to k-dimensional manifolds intrinsically without reference to an embedding in some S^n.

Here is a list of some computations of stable homotopy groups of spheres for you to reflect on. (Note: π_k^S has been computed for $k \leq 64$. There is no reasonalble conjecture for π_k^S for general k, although there are many results known. For example, in Chapter 10, we will show that the groups are finite for $k > 0; \pi_0^S = Z$ by the Hopf degree theorem.)

k	1	2	3	4	5	6
π_k^S	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/24$	0	0	$\mathbf{Z}/2$
k	7	8	9	10	11	12
π_k^S	$\mathbf{Z}/240$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^3$	$\mathbf{Z}/6$	$\mathbf{Z}/504$	0
k	13	14	15	16	17	18
π_k^S	$\mathbf{Z}/3$	$(\mathbf{Z}/2)^2$	$\mathbf{Z}/480 \oplus \mathbf{Z}/2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^4$	$\mathbf{Z}/8 \oplus \mathbf{Z}/2$
k	19	20	21	22	23	24
π_k^S	$\mathbf{Z}/264 \oplus \mathbf{Z}/2$	$\mathbf{Z}/24$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	†	$(\mathbf{Z}/2)^2$

† π_{23}^S *is* $Z/16 \oplus Z/8 \oplus Z/2 \oplus Z/9 \oplus Z/3 \oplus Z/5 \oplus Z/7 \oplus Z/13$.

The reference [32] is a good source for the tools to compute π_k^S. We will give stably framed manifolds representing π_k^S. for k <9; you may challenge your local homotopy theorist to supply the proofs. In this range there are (basically) two sources of framed manifolds:normal framings on spheres coming from the image of the stable J-homomorphism $J:\pi_k(O) \to \pi_k^S$, and tangential framing coming from Lie groups. There is considerable overlap between these sources.

Bott periodicity (Theorem 6.49) computes $\pi_k(O)$.

k	0	1	2	3	4	5	6	7	8
$\pi_k O$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}/2$

Then $J:\pi_k O \to \pi_k^S$ is an isomorphism for k=1, and epimorphism for k = 3, 7, and a monomorphism for k= 8.

Another source for framed manifolds are Lie groups. If G is a compact k-dimensional Lie group and $T_e G \cong R^k$ is an identification of its tangent space at the identity, then one can use the group multiplication of identify $TG \cong G \times R^k$ and thereby frame the tangent bundle. This is the so-called Lie invariant framing. The generators of the cyclic groups $\pi_0^S, \pi_1^S, \pi_2^S, \pi_3^S, \pi_4^S, \pi_5^S, \pi_6^S, \pi_7^S$ are given by e, $S^1, S^1 \times S^1, S^3, S^3 \times S^3, S^7$ with invariant framings. (The unit octonions S^7 fail to be a group because of the lack of associativity, but nonetheless, they do have an invariant framing.)

Finally, the generators of π_8^S are given by S^8 with framing given by the J-homomorphism and the unique exotic sphere in dimension 8. (An exotic sphere is a smooth manifold homeomorphic to a sphere and not diffeomorphic to a sphere.)

We have given a bordism description of the groups π_k^S. If X is any space $\pi_k^S X$ can be given a bordism description also. In this case one adds the structure of a map from the manifold to X. (A map from a manifold to a space X is sometimes called a singular manifold in X.)

Definition 7.16. Let $(V_i, \gamma_i : TV_i \oplus \Sigma^a \cong \Sigma^{k+a}), i = 0,1$ be two stably framed k-manifolds and $g_i : V_i \to X, i = 0,1$ two maps.

We say (V_0, γ_0, g_0) is stably framed bordant to (V_1, γ_1, g_1) over X if there exists a stably framed bordism (W, T) from (V_0, γ_0) to (V_0, γ_0) and a map

$$G: W \to X$$

Extending go and g1.

We introduce the notation:

1. Let X_+ denote X II pt, the union of X with a disjoint base point.
2. Let $\Omega_k^{fr}(X)$ denote the stably framed bordism classes of stably framed k-manifolds over X. Since every space maps uniquely to point, and since $S^0 = pt_+$, we can restate Corollary 7.15 in this notation as

$$\Omega_k^{fr}(pt) = \pi_k^S(pt_+)$$

Since every space maps uniquely to point, and since $S^0 = pt_+$, we can restate Corollary 7.15 in this notation as

$$\Omega_k^{fr}(pt) = \pi_k^S(pt_+)$$

Since $\pi_k^S = \pi_k^S(S^0) = \pi_k^S(pt_+)$.

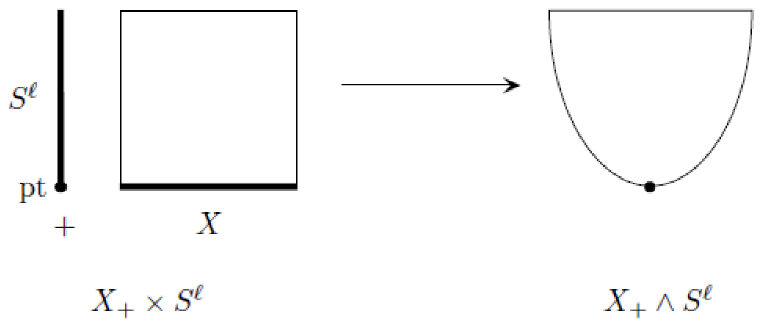
More generally one can easily prove the following theorem.

Therorem 7.17. $\Omega_k^{fr}(X) = \pi_k^S(X_+)$.

The proof of this theorem is essentially the same as for $X = pt$; one just has to carry the map $V \to X$ along for the ride. We give an outline of the argument and indicate a map $\pi_k^S(X_+) \to \Omega_k^{fr}(X)$.

Sketch of proof. Choose ℓ large so that $\pi_k^S(X_+) = \pi_{k+\ell}(X_+ \wedge S^\ell)$.

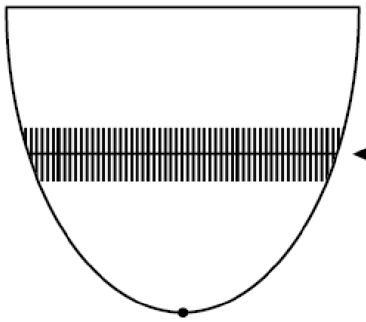
The smash product $X_+ \wedge S^\ell = S^\ell X_+ v S^\ell = X \times S^\ell / X \times pt$ is calles the half smash of X and S^ℓ and is depicted in the following picture.



Notes

Given $f: S^{k+\ell} \to X_+ \wedge S^\ell$, make f transverse to $X \times \{x\}$, where $x \Sigma S^\ell$ is a point different from the base point. (You should think carefully about what transversalitymeans since X is just a topological space. The point is that smoothness is only needed in the normal directions, since one can project to the sphere.)

Then $f^{-1}(X \times \{x\}) = V$ is a smooth, compact manifold, and since a neighborhood of $X \times \{x\} in X_+ \wedge S^\ell$ is homeomorphic to $X \times R^\ell$ as indicated in the following figure,



The submanifold V has a framed normal bundle, and

$f_{|V} : V \to X \times \{x\} = X$. This procedure shoes how to associate a stably framed manifold with a map to X to a (stable) map $f: S^{k+\ell} \to X_+ \wedge S^\ell$. One can show as before, using the Pontrjagin-Thom construction, that induced map $\pi_{k+\ell}(X_+ \wedge S^\ell) \to \Omega_k^{fr}(X)$ is an isomorphism.

Exercise : Define the reverse map $\Omega_k^{fr}(X) \to \pi_k^S(X_+)$.

Spectra

The collection of spheres, $\{S^n\}_{n=0}^{\infty}$, together with the maps (in fact homeomorphisms)

$$k_n : SS^n \xrightarrow{\cong} S^{n+1}$$

Forms a system of spaces and maps from which one can construct the stable homotopy groups $\pi_n^S(X)$. Another such system is the collection of Elienberg- MacLane spaces $\pi_n^S(X)$. from which we can recover the cohomology groups by the identification $H^n(X;Z)=\left[X,K(Z,n)\right]$s according the results of Chapter 7.

The notion of a spectrum abstracts from these two examples and introduces a category which measures "stable" phenomena, that is, phenomena which are preserved by suspending. Recall that $\tilde{H}^n$ and by definition $\pi_n^S(X)=\pi_{n+1}^S(SX)$. Thus cohomology and stable homotopy groups are measuring stable information about a space X.

Definition 7.18. A spectrum is a sequence of pairs $\{K_n,k_n\}$ where the K_n are based spaces and $k_n : SK_n \to K_{n+1}$ are basepoint preserving maps, where SK_n denotes the suspension.

In Exercises 95 you saw that the n-fold reduced suspension of S^nX of X is homeomorphic to S^nX Thus we can rewrite the definition of stable homotopy groups as

$$\pi_n^S X = \lim_{\ell\to\infty} \pi_{n+\ell}\left(S^\ell \wedge X\right)$$

Where the limit is taken over the homomorphisms

$$\pi_{n+\ell}\left(S^\ell \wedge X\right) \to \pi_{n+\ell+1}\left(S^{\ell+1} \wedge X\right).$$

Those homomorphisms are composites of the suspension

$$\pi_{n+\ell}\left(S^\ell \wedge X\right) \to \pi_{n+\ell+1}\left(S\left(S^{\ell+1} \wedge X\right)\right).$$

The identification $S\left(S^\ell \wedge X\right)=\mathrm{S}^1\wedge\left(S^\ell \wedge X\right)=S\left(S^\ell\right)\wedge \mathrm{X},.$ and the map $\pi_{n+\ell+1}\left(S\left(S^\ell\right)\wedge X\right)\to \pi_{n+\ell+1}\left(S^{\ell+1}\wedge X\right)$ induced by the map $k_\ell : S\left(S^\ell\right)\to S^{\ell+1}.$

Thus we see a natural link between the sphere spectrum

$$S=\left\{S^n, k_n : S\left(S^n\right)\cong S^{n+1}\right\}$$

And the stable homotopy groups

$$\pi_n^S(X) = \lim_{\ell\to\infty} \pi_{n+\ell}\left(S^\ell \wedge X\right).$$

Another example is provided by oridinary integral homology. The path space fibration and the long exact sequence in homotopy, shows that the loop space of the Eilenberg-MacLane spce $\pi_n^S(X) = \lim_{\ell\to\infty} \pi_{n+\ell}\left(S^\ell \wedge X\right).$ is homotopy equivalent to $K(Z,n).$ Fixing a model for V for each n, there exists a sequence of homotopy equivalences

$$h_n : K(Z,n) \to \Omega K(Z,\mathrm{n+1}).$$

Then h_n defines, by taking its adjoint, a map

$$k_n : \mathrm{S}\big(K(Z,n)\big) \to K(Z,\mathrm{n+1}).$$

In this way we obtain the Eilenbery-MacLane spectrum

$K(Z) = \{K(Z,n), \mathrm{k}_n\}.$

Have seen in Theorem 7.22 that $H^n(X;Z) = [X, K(Z,n)].$

Ordinary homology and cohomology are derived from the Eilenberg-MacLane spectrum, as the next theorem indicates. This point of view generalizes to motivate the definition of homology and cohomology with respect to any spectrum.

Theorem 7.19. For any space X,

1. $H_n(X,Z) = \lim_{n\to\ell} \pi_{n+\ell}\big(X_+ \wedge K(Z,\ell)\big).$
2. $H^n(X,Z) = \lim_{n\to\ell} \left[S^\ell(X_+), K(Z,n+\ell)\right]_0$

Recall that for $n \geq 0,$ $H^n(X) = \tilde{H}^n(X_+) = \tilde{H}^{n+1}(SX_+) = H^{n+1}(SX_+);$ in fact the diagram

$$\begin{array}{ccc} [X_+, K(\mathbf{Z}, n)]_0 & \xrightarrow{S} & [SX_+, SK(\mathbf{Z}, n)]_0 \\ \Big\downarrow{\scriptstyle h_n \cong} & & \Big\downarrow{\scriptstyle k_n} \\ [X_+, \Omega K(\mathbf{Z}, n+1)]_0 & \xrightarrow{\cong} & [SX_+, K(\mathbf{Z}, n+1)]_0 \end{array}$$

Commutes. This shows that we could have defined the cohomology of a space by

$$H^n(X;Z)\lim_{\ell\to\infty}\left[S^\ell X_+;K(Z,n+\ell)\right]_0,$$

And verifies the second part of this theorem. The first part can be proven by starting with this fact and using Spanier-Whitehead duality. See the project on Spanier-Whitehead duality at the end of this chapter.

These two examples and Theorem 7.19 leads to the following definition. Recall that X_+ denotes the space X with a disjoint base point. In particular, if $A \subset X$, then $(X_+ / A_+) = X / A$ if A is non-empty and equals X_+ if A is empty.

Definition 7.20. Let $K = \{K_n, K_n\}$ be a spectrum. Define the (unreduced) homology and cohomology with coefficients in the spectrum K to be the functor taking a space X to the abelian group

$$H_n(X;K) = \lim_{\ell\to\infty} \pi_{n+\ell}(X_+ \wedge K_\ell)$$

And

$$H_n(X;K) = \lim_{\ell\to\infty}\left[S^\ell(X_+);K_{n+\ell}\right]_0.$$

The reduced homology and cohomology with coefficients in the spectrum K to be the functor taking a based space X to the abelian group

$$\tilde{H}_n(X;K) = \lim_{\ell\to\infty} \pi_{n+\ell}(X \wedge K_\ell)$$

And

$$\tilde{H}_n(X;K) = \lim_{\ell\to\infty}\left[S^\ell \mathrm{X}; K_{n+\ell}\right]_0,$$

And the homology and cohomology of a pair with coefficients in the spectrum k to be the functor taking a pair of space (X,A) to the abelian group

$$H_n(X,A;K) = \lim_{\ell\to\infty} \pi_{n+\ell}\left(X_{+/A_+}(\wedge K_\ell)\right)$$

And

$$H_n(X,A;K) = \lim_{\ell\to\infty}\left[S^\ell \mathrm{X}(X_+/A_+)K_{n+\ell}\right]_0,$$

It is a theorem that these are generalized (co) homology theories; they satisfy all the Eilenberg-Steenrod axioms except the dimension axiom. We will discuss this is more detail later.

For example, stable homology theory $\tilde{H}_n(X;S) = \pi_n^S X$ is a reduced homology theory; framed bordism $H_n(X;S) = \pi_n^S X_+ = \Omega_n^{fr}(X)$ is an unreduced homology theory.

Note that $H_n(\mathrm{pt};\mathrm{K})$ can be non-zero for $n \neq 0,$ for example $H_n(\mathrm{pt};\mathrm{S}) = \pi_n^S$. Ordinary homologty is characterized by the fact that $H_n(\mathrm{pt}) = 0$ for $n \neq 0,$ (see Theorem 1.31). The groups $H_n(\mathrm{pt};\mathrm{K})$ are called the coefficients of the spectrum.

There are many relationships between homology, unreduced homology, suspension, and homology of pairs, some of which are obvious and some of which are not. We list some facts for homology.

- For a based space $X, \tilde{\mathrm{H}}_n(X;K) = \tilde{H}_{n+1}(SX;K).$
- For a space $X, \mathrm{H}_n(X;K) = \tilde{H}_n(X_+;K).$
- For a pair of spaces, $X, \mathrm{H}_n(X,A;K) \cong \tilde{H}_n(X/A;K).$
    - For a CW-pair, $\mathrm{H}_n(X,A;K)$ fats into the long exact sequence of a pair.

Check your progress

1. Prove: Let $V^k \subset S^n$ be a closed, oriented, normally framed sub-manifold of S^n. Then A normal framing $\gamma : \upsilon\left(V \to S^n\right) \cong \in^{n-k}$ induces a trivialization

$$\bar{\gamma} : TV \oplus \in^{n-k+1} \cong \in^{n+1} .$$

2. Prove: There is a 1-1 correspondence between stable tangential framings and stable normal framing of a manifold V. More precisely: Let $i : V \to S^n$ be an embedding. A stable framing of TV determines stable framing of $\upsilon(i)$ and conversely.

3. Explain about Suspension and Frudenthal theorem.

7. 5 LET US SUM UP

1. The collapse map induces a bijection $\Omega^{fr}_{k-n,M} \to \left[M, S^n\right]$.

This method of translating between bordism and homotopy sets is called the Pontrjagin-Thom construction.

2. Taking the suspension of a map corresponds, via the Pontrjagin-Thom construction, to the same manifold V, but embedded in the equation $S^k \subset S^{k+1}$, and with normal framing the direct sum of the old normal framing and the trivial 1-dimensional framing.

3. Let $V^k \subset S^n$ be a closed, oriented, normally framed sub-manifold of S^n. Then

A normal framing $\gamma:\upsilon\left(V\to S^{n}\right)\cong\in^{n-k}$ induces a trivialization $\bar{\gamma}:TV\oplus\in^{n-k+1}\cong\in^{n+1}$.

7. 6 KEY WORDS

Framed bordism

Homotophy groups of spheres

Eilenberg-MacLane spce

Cohomology

Tangential framings

7.7 QUESTIONS FOR REVIEW

1. Explain about homotopy groups of spheres
2. Explain about tangential framings
3. Explain about Frudenthal theorem

7.8 SUGGESTIVE READINGS AND REFERENCES

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7.9 ANSWER TO CHECK YOUR PROGRESS

1. See section 7.2
2. See section 7.4
3. See section 7.3